

# Stability analysis of the operational space control for industrial robots using their own joint velocity PI controllers

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## SUMMARY

Operational space control of industrial robots is addressed in this document. We analyze a two-loop hierarchical control with the resolved motion rate controller (RMRC) as outer loop and the joint velocity PI controller as inner loop; the latter is the typical velocity controller used in industrial robots. We prove, by the first time, that these simple controllers make the solutions of the closed-loop system uniformly ultimately bounded. Additionally, we give some simple guidelines for the selection of the control gains so as to ensure an explicit bound of the tracking error.

**KEYWORDS:** stability analysis; Industrial robots; operational space control; Hierarchical control.

## 1. Introduction

Since their appearance in the early 1960s, industrial robots have gained wide popularity as essential components in the construction of automated systems. Reduction of manufacturing costs, increase in productivity, improvement of product quality standards, and the possibility of eliminating harmful or repetitive tasks for human operators represent the main factors that have determined the spreading of the robotics technology in the manufacturing industry.<sup>1</sup> Industrial robots are suitable for applications where high precision, repeatability and tracking accuracy are required. These facts give a great importance to the stability analysis of the actual control schemes of industrial robots.

It is common to specify the robotic tasks in terms of the pose of the robot's end-effector. The operational space, introduced by Khatib,<sup>2</sup> considers the description of the pose of the robot's end-effector based on a position vector, given in Cartesian coordinates, and an orientation vector, specified in terms of Euler angles.

It is well-known that most of industrial robots have internal joint velocity PI controllers, which are usually tuned with very high proportional and integral gains. In practice, these internal controllers are fixed, and an outer loop is necessary to achieve the control goal in operational space. This two-loop hierarchical structure is shown in Fig. 1.

Aicardi *et al.*<sup>3</sup> were the first in analyzing such a hierarchical structure to solve the problem of pose control; they employed a velocity controller based on an inverse dynamics methodology.<sup>4</sup> More recently, Kelly and Moreno<sup>5</sup> used the same inner controller, together with the so-called resolved motion rate control (RMRC),<sup>6</sup> to solve the problem of operational space control.

The main idea of this paper is to analyze the stability of the motion control in operational space of industrial robots, considering their actual internal joint velocity PI controllers, plus the RMRC as the outer loop. Then, by using the Lyapunov theory we conclude, for the first time, that the solutions of the closed-loop system are uniformly ultimately bounded.<sup>7</sup>

There are some previous works which have been bases for our main contribution; such works (see ref. [8–11]) use a linear PD feedback control for the trajectory tracking of robot motion in joint space. Kawamura *et al.*<sup>8</sup> and Wang and Chen<sup>9</sup> proved that high gains make the tracking error uniformly bounded. Furthermore, Qu and Dorsey,<sup>10</sup> and Corless and Leitmann<sup>11</sup> proved that the solutions of the closed-loop system are uniformly ultimately bounded.

We invoke the results of Qu and Dorsey<sup>10</sup> in order to apply them in the analysis of the inner joint velocity PI controller. Then, we complete the overall stability analysis of motion control in operational space by using the RMRC control as the kinematic outer-loop controller. To the best of the authors' knowledge, the analysis proposed here is the first one that considers the actual loops contained in industrial robots.

The content of this document is organized as follows. In Section 2 we introduce the dynamic and kinematic models of the robot. In Section 3 we present the description of the outer kinematic control, the inner joint velocity PI controller, and the overall closed-loop system. The complete stability analysis is given in Section 4; a tuning procedure is also presented in this section. Finally, the conclusions of our work are given in Section 5.

Some remarks on notation: We use  $\lambda_{\min}\{A\}$  and  $\lambda_{\max}\{A\}$  to represent, respectively, the smallest and the largest eigenvalues of a symmetric positive definite matrix  $A(\mathbf{y})$ , for any  $\mathbf{y} \in \mathbb{R}^n$ . Given  $\mathbf{y} \in \mathbb{R}^n$  and a matrix  $A(\mathbf{y})$  the Euclidean norm of  $\mathbf{y}$  is defined as  $\|\mathbf{y}\| = \sqrt{\mathbf{y}^T \mathbf{y}}$ , and the induced norm of  $A(\mathbf{y})$  is defined as  $\|A(\mathbf{y})\| = \sqrt{\lambda_{\max}\{A^T A\}}$ . Given a

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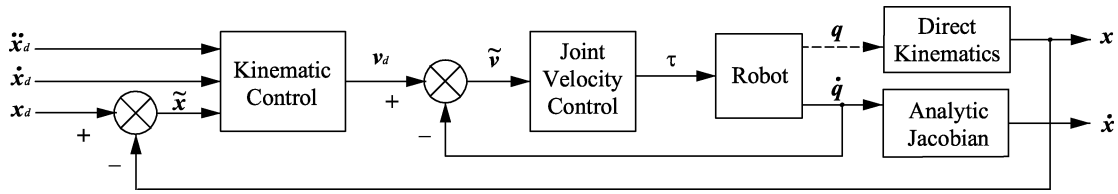


Fig. 1. Typical scheme of a hierarchical structure for operational space control.

symmetric matrix  $A(\mathbf{u})$  and  $\mathbf{u}, \mathbf{w}, \mathbf{y} \in \mathbb{R}^n$ , we have

$$\lambda_{\min}\{A\}\|\mathbf{y}\|^2 \leq \mathbf{y}^T A(\mathbf{u})\mathbf{y} \leq \lambda_{\max}\{A\}\|\mathbf{y}\|^2, \quad (1)$$

$$\pm A(\mathbf{u})\mathbf{w} \leq \lambda_{\max}\{A\}\|\mathbf{w}\|, \quad (2)$$

$$\pm \mathbf{y}^T A(\mathbf{u})\mathbf{w} \leq \lambda_{\max}\{A\}\|\mathbf{y}\|\|\mathbf{w}\|. \quad (3)$$

## 2. Robot Model

### 2.1. Dynamic model

The dynamic model of a serial  $n$ -link robot free of friction and elasticity is written as<sup>12</sup>

$$M(\mathbf{q})\ddot{\mathbf{q}} + C(\mathbf{q}, \dot{\mathbf{q}})\dot{\mathbf{q}} + \mathbf{g}(\mathbf{q}) = \boldsymbol{\tau}, \quad (4)$$

where  $\mathbf{q} \in \mathbb{R}^n$  is the vector of joint coordinates,  $\dot{\mathbf{q}} \in \mathbb{R}^n$  is the vector of joint velocities,  $\ddot{\mathbf{q}} \in \mathbb{R}^n$  is the vector of joint accelerations,  $\boldsymbol{\tau} \in \mathbb{R}^n$  is the vector of applied torque inputs,  $M(\mathbf{q}) \in \mathbb{R}^{n \times n}$  is the symmetric positive definite manipulator inertia matrix,  $C(\mathbf{q}, \dot{\mathbf{q}}) \in \mathbb{R}^{n \times n}$  is the matrix of centripetal and Coriolis torques and  $\mathbf{g}(\mathbf{q}) \in \mathbb{R}^n$  is the vector of gravitational torques.

Some useful properties of the dynamic model (4) that will be used in this document, are the following<sup>12</sup>:

**Property 1:**  $C(\mathbf{q}, \dot{\mathbf{q}})$  can be chosen so that matrix  $\frac{1}{2}\dot{M}(\mathbf{q}) - C(\mathbf{q}, \dot{\mathbf{q}})$  is skew-symmetric, i.e.,

$$\mathbf{y}^T \left[ \frac{1}{2}\dot{M}(\mathbf{q}) - C(\mathbf{q}, \dot{\mathbf{q}}) \right] \mathbf{y} = 0, \quad \forall \mathbf{y} \in \mathbb{R}^n. \quad (5)$$

**Property 2:** There exist positive constants  $k_M, k_C$  and  $k_g$  such that, for all  $\mathbf{u}, \mathbf{w}, \mathbf{y} \in \mathbb{R}^n$ , we have

$$\|M(\mathbf{u})\mathbf{y}\| \leq k_M\|\mathbf{y}\|, \quad (6)$$

$$\|C(\mathbf{u}, \mathbf{w})\mathbf{y}\| \leq k_C\|\mathbf{w}\|\|\mathbf{y}\|, \quad (7)$$

$$\|\mathbf{g}(\mathbf{u})\| \leq k_g. \quad (8)$$

### 2.2. Kinematic model

The direct kinematic model of the robot can be written as

$$\mathbf{x} = \mathbf{h}(\mathbf{q}), \quad (9)$$

where  $\mathbf{h}(\mathbf{q}) : \mathbb{R}^n \rightarrow \mathbb{R}^6$ , with  $n \geq 6$ , is a function describing the relation between the vector of joint coordinates  $\mathbf{q} \in \mathbb{R}^n$  and the end-effector's pose (position and orientation), which is given by  $\mathbf{x} \in \mathbb{R}^6$ , that is,  $\mathbf{x}$  belongs to the operational space. The differential kinematics equation can be obtained as the time derivative of the direct kinematics Eq. (9), i.e.,<sup>1</sup>

$$\dot{\mathbf{x}} = J(\mathbf{q})\dot{\mathbf{q}}, \quad (10)$$

where  $J(\mathbf{q}) \in \mathbb{R}^{6 \times n}$  is the so-called analytic Jacobian matrix. In the forthcoming analysis, it is assumed that the robot's end-effector motion makes  $J(\mathbf{q})$  to be full rank during the whole task. Also,  $\|J(\mathbf{q})\|$  is assumed to be bounded by a positive constant  $k_J$ , i.e.,

$$\|J(\mathbf{q})\| \leq k_J \quad \forall \mathbf{q} \in \mathbb{R}^n. \quad (11)$$

The right pseudoinverse of  $J(\mathbf{q})$  is given by<sup>1</sup>

$$J(\mathbf{q})^\dagger = J(\mathbf{q})^T [J(\mathbf{q})J(\mathbf{q})^T]^{-1}, \quad (12)$$

in such a way that  $J(\mathbf{q})J(\mathbf{q})^\dagger = I$ , being  $I \in \mathbb{R}^{6 \times 6}$  the identity matrix. As  $J(\mathbf{q})$  is full rank, then, also  $J(\mathbf{q})^\dagger$  is full rank, satisfying

$$\|J(\mathbf{q})^\dagger\| \leq k_{Jp}, \quad \forall \mathbf{q} \in \mathbb{R}^n, \quad (13)$$

and its time derivative

$$\| \dot{J}(\mathbf{q})^\dagger \| = \left\| \frac{d}{dt} \{J(\mathbf{q})^\dagger\} \right\| \leq k_{dJp}, \quad \forall \mathbf{q} \in \mathbb{R}^n, \quad (14)$$

where  $k_{Jp}$  and  $k_{dJp}$  are positive constants.

## 3. Hierarchical Controller

### 3.1. Kinematic control

By kinematic control we refer to any scheme that uses an inverse Jacobian algorithm to resolve the desired joint velocities directly from the pose variables of the desired task. Thus, a kinematic controller is used as the outer loop of a hierarchical controller, which provides the desired joint velocities for the inner velocity controller (Fig. 1).

In this paper we use as kinematic controller the RMRC, which was first proposed by Whitney and Holzkecht.<sup>6</sup> Using this scheme, the desired joint velocity for the inner loop, can be written as

$$\mathbf{v}_d = J(\mathbf{q})^\dagger [\dot{\mathbf{x}}_d + K\tilde{\mathbf{x}}], \quad (15)$$

where  $\mathbf{v}_d \in \mathbb{R}^n$  is the desired joint velocity vector,  $\dot{\mathbf{x}}_d \in \mathbb{R}^6$  is the time derivative of the desired pose vector  $\mathbf{x}_d$  in operational space,  $\tilde{\mathbf{x}} = \mathbf{x}_d - \mathbf{x} \in \mathbb{R}^6$  is the pose error vector in operational space and  $K \in \mathbb{R}^{6 \times 6}$  is a symmetric positive definite matrix of control gains. Under the assumption of perfect velocity tracking, that is,  $\dot{\mathbf{q}} \equiv \mathbf{v}_d$ , pre-multiplying both sides of (15) by  $J(\mathbf{q})$  and substituting (10), we can write

$$\dot{\tilde{\mathbf{x}}} = -K\tilde{\mathbf{x}}. \quad (16)$$

As  $K$  is a symmetric positive definite matrix, we conclude that  $\tilde{x} \rightarrow 0$  as  $t \rightarrow \infty$ . However, a real joint velocity controller does not ensure the instantaneous tracking of the desired velocity  $v_d$ . In practice, we get asymptotic velocity tracking instead of ideal velocity tracking. So, the implementation of the kinematic control (15) requires the design of a joint velocity controller. To this end, let us define the joint velocity error as

$$\tilde{v} = v_d - \dot{q} \in \mathbb{R}^n. \quad (17)$$

From (10), (15) and (17), we get

$$\dot{\tilde{x}} = -K\tilde{x} + J(q)\tilde{v}, \quad (18)$$

instead of (16).

In this work, we prove that the practical implementation of (15) and the intrinsic joint velocity PI inner loop of industrial robots guarantee the uniform ultimate boundedness of the joint velocity error  $\tilde{v}$ , and the pose error in operational space  $\tilde{x}$ , for all initial conditions  $\tilde{x}(t_0)$  and  $\tilde{v}(t_0)$ , i.e.,

$$\lim_{t \rightarrow \infty} \left\{ \begin{array}{l} \|\tilde{x}(t)\| \\ \|\tilde{v}(t)\| \end{array} \right\} \leq b \quad (19)$$

where  $b$  is a sufficiently small constant.

For the analysis, we also need the time derivative of (15) which is given by

$$\dot{v}_d = \dot{J}(q)^\dagger[\dot{x}_d + K\tilde{x}] + J(q)^\dagger[\ddot{x}_d + K\dot{\tilde{x}}], \quad (20)$$

where  $\ddot{x}_d \in \mathbb{R}^6$  is the vector of desired operational acceleration, and  $\dot{\tilde{x}} \in \mathbb{R}^6$  is the velocity error in operational space, given by (18). We assume that  $x_d$ ,  $\dot{x}_d$  and  $\ddot{x}_d$  are bounded, and we denote by  $\|x_d\|_M$ ,  $\|\dot{x}_d\|_M$ ,  $\|\ddot{x}_d\|_M$  the supreme values over the norms  $\|x_d\|$ ,  $\|\dot{x}_d\|$  and  $\|\ddot{x}_d\|$ , respectively, for all time  $t \geq t_0$ , that is,

$$x_d \leq \|x_d\|_M, \quad \dot{x}_d \leq \|\dot{x}_d\|_M, \quad \ddot{x}_d \leq \|\ddot{x}_d\|_M. \quad (21)$$

### 3.2. Joint velocity PI controller

For the inner loop in Fig. 1, we consider the classical joint velocity proportional integral PI controller commonly used in industrial robots, which can be written as

$$\tau = K_p \tilde{v} + K_i \xi, \quad (22)$$

$$\dot{\xi} = \tilde{v}, \quad (23)$$

where  $K_p, K_i \in \mathbb{R}^{n \times n}$  are diagonal positive definite matrices,  $\tilde{v}$  is defined in (17), and  $\xi = \int_0^t \tilde{v}(\sigma) d\sigma$ .

Without loss of generality, let us take

$$K_p = [k_p + \gamma] I, \quad (24)$$

$$K_i = \alpha K_p, \quad (25)$$

where  $k_p, \gamma$  and  $\alpha$  are strictly positive constants. Notice that (25) can be written as

$$K_i = [k_i + \alpha\gamma] I, \quad (26)$$

with  $k_i = \alpha k_p$ , or

$$\alpha = \frac{k_i}{k_p}. \quad (27)$$

### 3.3. Closed-loop system

Note in Fig. 1 that the closed-loop system is formed by the RMRC controller (15), the joint velocity PI controller (22)–(23) and the robot model given by (4), (9) and (10).

Substituting (22) into the robot dynamics (4) we have

$$M(q)\ddot{q} + C(q, \dot{q})\dot{q} + g(q) = K_p \tilde{v} + K_i \xi. \quad (28)$$

Now, taking into account (17) and its derivative, we get

$$\begin{aligned} M(q)\dot{\tilde{v}} + C(q, \dot{q})\tilde{v} + K_p \tilde{v} + K_i \xi \\ = M(q)\dot{v}_d + C(q, \dot{q})v_d + g(q). \end{aligned} \quad (29)$$

We can add the terms  $\alpha M(q)\tilde{v} + \alpha C(q, \dot{q})\xi$  to both sides of the last equation to obtain

$$\begin{aligned} M(q)[\dot{\tilde{v}} + \alpha\tilde{v}] + C(q, \dot{q})[\tilde{v} + \alpha\xi] + K_p \tilde{v} + K_i \xi \\ = M(q)[\dot{v}_d + \alpha\tilde{v}] + C(q, \dot{q})[v_d + \alpha\xi] + g(q). \end{aligned} \quad (30)$$

Finally, from (18), (23) and (30), we get the closed-loop system in terms of the state vector  $z = [\tilde{x}^T \quad \xi^T \quad \tilde{v}^T]^T \in \mathbb{R}^{6+2n}$ :

$$\begin{aligned} \frac{d}{dt} \begin{bmatrix} \tilde{x} \\ \xi \\ \tilde{v} \end{bmatrix} \\ = \begin{bmatrix} J(q)\tilde{v} - K\tilde{x} \\ \tilde{v} \\ -\alpha\tilde{v} - M(q)^{-1}[C(q, \dot{q})[\tilde{v} + \alpha\xi] + K_p \tilde{v} + K_i \xi - d(t, z)] \end{bmatrix}, \end{aligned} \quad (31)$$

where the term

$$d(t, z) = M(q)[\dot{v}_d + \alpha\tilde{v}] + C(q, \dot{q})[v_d + \alpha\xi] + g(q) \quad (32)$$

will be considered as a disturbance and, as shown in the Appendix A, it is upper bounded for all  $t \geq t_0$  by a quadratic polynomial:

$$\|d(t, z)\| \leq \varsigma_2 \|z(t)\|^2 + \varsigma_1 \|z(t)\| + \varsigma_0, \quad (33)$$

where  $\varsigma_0, \varsigma_1, \varsigma_2$  are positive constants.

Substituting the definitions (24) and (26) in (31), we get

$$\begin{aligned} \frac{d}{dt} \begin{bmatrix} \tilde{x} \\ \xi \\ \tilde{v} \end{bmatrix} \\ = \begin{bmatrix} J(q)\tilde{v} - K\tilde{x} \\ \tilde{v} \\ -\alpha\tilde{v} - M(q)^{-1}[C(q, \dot{q})[\tilde{v} + \alpha\xi] + k_p[\tilde{v} + \alpha\xi] + \gamma[\tilde{v} + \alpha\xi] - d(t, z)] \end{bmatrix}. \end{aligned} \quad (34)$$

In accordance with the theory of perturbed systems<sup>7</sup> and for future ease of reference, we define

$$\begin{aligned} & \frac{d}{dt} \begin{bmatrix} \bar{x} \\ \xi \\ \bar{v} \end{bmatrix} \\ &= \begin{bmatrix} J(\mathbf{q})\bar{v} - K\bar{x} \\ \bar{v} \\ -\alpha\bar{v} - M(\mathbf{q})^{-1} [C(\mathbf{q}, \dot{\mathbf{q}}) [\bar{v} + \alpha\xi] + k_p [\bar{v} + \alpha\xi]] \end{bmatrix} \end{aligned} \quad (35)$$

as the nominal system from the closed-loop system (34). As we can see, (35) is free of disturbances and the origin of the state space  $\mathbf{z} = \mathbf{0}$  is an equilibrium of (35).

#### 4. Stability Analysis

In order to show the stability of the overall closed-loop system (34), we follow the methodology proposed by Khalil,<sup>7</sup> that is:

- First, to prove the asymptotic stability of the nominal system (35).
- Then, to prove the uniform ultimate boundedness of the solutions of the overall closed-loop system (34).

##### 4.1. Stability of the nominal system

To prove that the origin of the nominal system (35) is exponentially stable, we propose the following Lyapunov function candidate inspired from Qu and Dorsey:<sup>10</sup>

$$\begin{aligned} V(t, \mathbf{z}) &= \frac{1}{2} \bar{x}^T \bar{x} + \alpha \xi^T \left[ k_p I + \frac{1}{2} \alpha M(\mathbf{q}) \right] \xi \\ &\quad + \alpha \xi^T M(\mathbf{q}) \bar{v} + \frac{1}{2} \bar{v}^T M(\mathbf{q}) \bar{v}. \end{aligned} \quad (36)$$

It is easy to show that

$$\begin{bmatrix} \|\bar{x}\| \\ \|\xi\| \\ \|\bar{v}\| \end{bmatrix}^T P_1 \begin{bmatrix} \|\bar{x}\| \\ \|\xi\| \\ \|\bar{v}\| \end{bmatrix} \leq V(t, \mathbf{z}) \leq \begin{bmatrix} \|\bar{x}\| \\ \|\xi\| \\ \|\bar{v}\| \end{bmatrix}^T P_2 \begin{bmatrix} \|\bar{x}\| \\ \|\xi\| \\ \|\bar{v}\| \end{bmatrix},$$

with

$$\begin{aligned} P_1 &= \begin{bmatrix} \frac{1}{2} & 0 & 0 \\ 0 & \alpha k_p + \frac{1}{2} \alpha^2 \lambda_{\min}\{M\} & -\frac{1}{2} \alpha \lambda_{\max}\{M\} \\ 0 & -\frac{1}{2} \alpha \lambda_{\max}\{M\} & \frac{1}{2} \lambda_{\min}\{M\} \end{bmatrix}, \\ P_2 &= \begin{bmatrix} \frac{1}{2} & 0 & 0 \\ 0 & \alpha k_p + \frac{1}{2} \alpha^2 \lambda_{\max}\{M\} & \frac{1}{2} \alpha \lambda_{\max}\{M\} \\ 0 & \frac{1}{2} \alpha \lambda_{\max}\{M\} & \frac{1}{2} \lambda_{\max}\{M\} \end{bmatrix}, \end{aligned}$$

so that, the Lyapunov function candidate (36) satisfies

$$\lambda_1 \|\mathbf{z}\|^2 \leq V(t, \mathbf{z}) \leq \lambda_2 \|\mathbf{z}\|^2, \quad (37)$$

for some positive constants  $\lambda_1, \lambda_2$ , given by

$$\lambda_1 = \lambda_{\min}\{P_1\}, \quad \lambda_2 = \lambda_{\max}\{P_2\}. \quad (38)$$

Notice that  $V(t, \mathbf{z})$  is a globally positive definite function if

$$\alpha \leq \frac{2k_p \lambda_{\min}\{M\}}{\lambda_{\max}^2\{M\} - \lambda_{\min}^2\{M\}}, \quad (39)$$

or, using (27), if

$$k_i \leq \frac{2k_p^2 \lambda_{\min}\{M\}}{\lambda_{\max}^2\{M\} - \lambda_{\min}^2\{M\}}. \quad (40)$$

On the other hand, the time derivative of the Lyapunov function candidate (36) is given by

$$\begin{aligned} \dot{V}(t, \mathbf{z}) &= \bar{x}^T \dot{\bar{x}} + \alpha k_p \xi^T \dot{\xi} + \alpha^2 \xi^T M(\mathbf{q}) \dot{\xi} \\ &\quad + \frac{1}{2} \alpha^2 \xi^T \dot{M}(\mathbf{q}) \xi + \alpha \dot{\xi}^T M(\mathbf{q}) \bar{v} + \alpha \xi^T \dot{M}(\mathbf{q}) \bar{v} \\ &\quad + \alpha \xi^T M(\mathbf{q}) \dot{\bar{v}} + \bar{v}^T M(\mathbf{q}) \dot{\bar{v}} + \frac{1}{2} \bar{v}^T \dot{M}(\mathbf{q}) \bar{v}, \end{aligned} \quad (41)$$

which along the solutions of the nominal system (35) results in

$$\dot{V}(t, \mathbf{z}) = -\bar{x}^T K \bar{x} - \alpha k_i \xi^T \xi - k_p \bar{v}^T \bar{v} + \bar{x}^T J(\mathbf{q}) \bar{v},$$

where we have used the definition of  $\alpha$  in (27), and the skew-symmetry property (5) to eliminate some terms.

Then, under the assumption that the Jacobian matrix  $J(\mathbf{q})$  is bounded, see (11), it is easy to show that

$$\dot{V}(t, \mathbf{z}) \leq - \begin{bmatrix} \|\bar{x}\| \\ \|\xi\| \\ \|\bar{v}\| \end{bmatrix}^T Q \begin{bmatrix} \|\bar{x}\| \\ \|\xi\| \\ \|\bar{v}\| \end{bmatrix}, \quad (42)$$

with

$$Q = \begin{bmatrix} \lambda_{\min}\{K\} & 0 & -\frac{1}{2} k_J \\ 0 & \alpha k_i & 0 \\ -\frac{1}{2} k_J & 0 & k_p \end{bmatrix}. \quad (43)$$

So that (42) can be written as

$$\dot{V}(t, \mathbf{z}) \leq -\lambda_3 \|\mathbf{z}\|^2, \quad (44)$$

where

$$\lambda_3 = \lambda_{\min}\{Q\}. \quad (45)$$

Notice that we can choose

$$\lambda_{\min}\{K\} > \frac{k_j^2}{4k_p}, \quad (46)$$

so that  $Q$  is positive definite and  $\dot{V}(t, \mathbf{z})$  is globally negative definite.

Thus, provided that (40) and (46) are satisfied, and according to Lyapunov's direct method,<sup>12</sup> we conclude the exponential stability of the origin of the nominal system (35). This implies that the origin of the state space is attractive, i.e.,

$$\lim_{t \rightarrow \infty} \mathbf{z}(t) = \lim_{t \rightarrow \infty} \begin{bmatrix} \tilde{\mathbf{x}}(t) \\ \tilde{\boldsymbol{\xi}}(t) \\ \tilde{\mathbf{v}}(t) \end{bmatrix} = \mathbf{0}.$$

#### 4.2. Stability of the overall closed-loop system

We have shown in the previous subsection that the equilibrium of the nominal system (35) is exponentially stable by choosing the feedback gains  $k_i$ , and  $\lambda_{\min}\{K\}$  according to (40) and (46), respectively. Now, we are able to show that, by properly choosing the feedback gain  $\gamma$  of the joint velocity PI controller, the solutions  $\mathbf{z}(t)$  of the overall closed-loop system (34) are uniformly ultimately bounded.

First, we recall the following fact,<sup>10</sup> that will be useful for our purposes.

**Fact 1:** Let

$$g(\|\mathbf{z}\|) = -\alpha_1 \|\mathbf{z}\|^2 + \alpha_2 \|\mathbf{z}\| + \alpha_3, \quad (47)$$

be a quadratic polynomial with  $\alpha_1 > 0$ , and  $\alpha_2, \alpha_3 \geq 0$ . Given

$$\eta = \frac{\alpha_2 + \sqrt{\alpha_2^2 + 4\alpha_1\alpha_3}}{2\alpha_1}, \quad (48)$$

then

- $g(\eta) = 0$ ,
- $g(\|\mathbf{z}\|) < 0$ , and strictly decreasing, for all  $\|\mathbf{z}\| > \eta$ .

◇

Now, we introduce a variation of an existing theorem (see Theorem 4.18 in ref. [7]) which is inspired in the results of Qu and Dorsey<sup>10</sup> and Corless and Leitmann.<sup>11</sup>

**Theorem 1:** Let

$$\dot{\mathbf{z}} = \mathbf{f}(t, \mathbf{z}), \quad (49)$$

be a system that describes a first-order vector differential equation, where  $\mathbf{f} : [0, \infty) \times D \rightarrow \mathbb{R}^m$  and  $D \subset \mathbb{R}^m$  is a domain that contains the origin. Let  $V(t, \mathbf{z})$ , with  $V : [0, \infty) \times D \rightarrow \mathbb{R}$ , be a Lyapunov-like function that satisfies

$$\lambda_1 \|\mathbf{z}\|^2 \leq V(t, \mathbf{z}) \leq \lambda_2 \|\mathbf{z}\|^2, \quad (50)$$

$$\dot{V}(t, \mathbf{z}) \leq g(\|\mathbf{z}\|) < 0, \quad \forall \|\mathbf{z}\| > \eta, \quad (51)$$

for all  $t \geq t_0$ , and for all  $\mathbf{z} \in D$ , with  $\lambda_1, \lambda_2 > 0$ ,  $g(\|\mathbf{z}\|)$  and  $\eta$  defined in Fact 1 by (47) and (48), respectively. Then, there exists  $T \geq 0$  (dependent on  $\mathbf{z}(t_0)$  and  $\eta$ ) such that the solution of (49) satisfies the following properties, for any initial state  $\mathbf{z}_0 = \mathbf{z}(t_0)$ :

A. *Uniform boundedness*

$$\|\mathbf{z}(t)\| \leq \sqrt{\frac{\lambda_2}{\lambda_1}} \max\{\|\mathbf{z}_0\|, \eta\}, \quad \forall t \geq t_0, \quad (52)$$

B. *Uniform ultimate boundedness*

$$\|\mathbf{z}(t)\| \leq \sqrt{\frac{\lambda_2}{\lambda_1}} \eta', \quad \forall t \geq t_0 + T, \quad (53)$$

for all  $\eta' > \eta$ .

▽

The proof of Theorem 1 is based in a similar one presented by Corless and Leitmann,<sup>11</sup> and is given in Appendix B.

We are now ready to introduce our main result, which is inspired from Qu and Dorsey.<sup>10</sup>

**Proposition 1:** Consider the overall closed-loop system (34) with the outer-loop kinematic controller

$$\mathbf{v}_d = J(\mathbf{q})^\dagger [\dot{\mathbf{x}}_d + K\tilde{\mathbf{x}}], \quad (54)$$

and the inner-loop velocity controller

$$\boldsymbol{\tau} = K_p \tilde{\mathbf{v}} + K_i \tilde{\boldsymbol{\xi}} = [k_p + \gamma] \tilde{\mathbf{v}} + [k_i + \gamma\alpha] \tilde{\boldsymbol{\xi}}. \quad (55)$$

Suppose the control gains are chosen so that  $k_p$ ,  $k_i$  and  $K$  satisfy (40) and (46). Moreover,  $\gamma$  is a positive scalar such that

$$\gamma \geq \frac{1}{\epsilon} [\sigma^2 \varsigma_2 + \sigma \varsigma_1 + \varsigma_0], \quad (56)$$

where  $\varsigma_0$ ,  $\varsigma_1$  and  $\varsigma_2$  are the coefficients of the upper bound of  $\|\mathbf{d}(t, \mathbf{z})\|$ , given in (33),  $\epsilon$  is a positive constant such that

$$0 < \epsilon < \frac{\lambda_3}{\varsigma_2}, \quad (57)$$

and

$$\sigma = \sqrt{\frac{\lambda_2}{\lambda_1}} \max\{\|\mathbf{z}(t_0)\|, \bar{\eta}\}, \quad (58)$$

with

$$\bar{\eta} = \frac{\epsilon \varsigma_1 + \sqrt{\epsilon^2 \varsigma_1^2 + 4\epsilon \varsigma_0 [\lambda_3 - \epsilon \varsigma_2]}}{2[\lambda_3 - \epsilon \varsigma_2]}, \quad (59)$$

where  $\lambda_1, \lambda_2$ , are defined in (38), and  $\lambda_3$  is in (45).

Then, the system is stabilizable in the sense that exists  $T \geq 0$  such that the solution  $\mathbf{z}(t)$ , with initial state  $\mathbf{z}_0 = \mathbf{z}(t_0)$

satisfies

$$\|z(t)\| \leq \sigma, \quad \forall t \geq t_0, \quad (60)$$

$$\|z(t)\| \leq \sqrt{\frac{\lambda_2}{\lambda_1}} \bar{\eta}', \quad \forall t \geq t_0 + T, \quad (61)$$

for all  $\bar{\eta}' > \bar{\eta}$ , where (60) and (61) regard to the uniform and uniform ultimate boundedness, respectively.

▽

**Proof:** We follow the steps in the proof of Theorem 2 in ref. [10].

First, let us consider again the Lyapunov function candidate  $V(t, z)$  defined in (36) and satisfying (50). Taking the time derivative along the trajectories of the overall closed-loop system (34), we have that

$$\dot{V}(t, z) \leq -\lambda_3 \|z\|^2 - \gamma \varphi(t, z)^T \varphi(t, z) + \varphi(t, z)^T d(t, z),$$

where  $\varphi(t, z) = \alpha \xi + \bar{v}$ . Thus

$$\dot{V}(t, z) \leq -\lambda_3 \|z\|^2 - \gamma \|\varphi(t, z)\|^2 + \|\varphi(t, z)\| \|d(t, z)\|. \quad (62)$$

In order to find an upper bound for  $\dot{V}(t, z)$ , we first consider that  $\|\varphi(t, z)\| < \epsilon$ . In such a case, as  $\gamma > 0$ , we can write (62) as

$$\dot{V}(t, z) \leq -\lambda_3 \|z\|^2 + \epsilon \|d(t, z)\|,$$

and using (33), we get

$$\dot{V}(t, z) \leq -[\lambda_3 - \epsilon \zeta_2] \|z\|^2 + \epsilon \zeta_1 \|z\| + \epsilon \zeta_0. \quad (63)$$

Now consider the case  $\|\varphi(t, z)\| \geq \epsilon$  and let us define  $\delta = \sqrt{\frac{\lambda_2}{\lambda_1}}$  so that, from (50), we have  $\lambda_2 \geq \lambda_1$  and  $\delta \geq 1$ . Also, from the definition of  $\sigma$  (58) we have that  $\sigma \geq \|z(t_0)\|$ , and thus from (56)

$$\gamma \geq \frac{1}{\epsilon} [\delta^2 \|z(t_0)\|^2 \zeta_2 + \delta \|z(t_0)\| \zeta_1 + \zeta_0]. \quad (64)$$

Considering (62), (64) and (33), we have

$$\begin{aligned} \dot{V}(t, z) &\leq -\lambda_3 \|z\|^2 - \gamma \|\varphi(t, z)\| + \|\varphi(t, z)\| \|d(t, z)\|, \\ &\leq -\lambda_3 \|z\|^2 - \|\varphi(t, z)\| [\delta^2 \|z(t_0)\|^2 \zeta_2 + \delta \|z(t_0)\| \zeta_1 \\ &\quad + \zeta_0] + \|\varphi(t, z)\| [\|z(t)\|^2 \zeta_2 + \|z(t)\| \zeta_1 + \zeta_0], \\ &\leq -\lambda_3 \|z\|^2 + \|\varphi(t, z)\| [(\|z(t)\|^2 - \delta^2 \|z(t_0)\|^2) \zeta_2 \\ &\quad + (\|z(t)\| - \delta \|z(t_0)\|) \zeta_1]. \end{aligned} \quad (65)$$

Notice that, starting at time  $t_0$ , we can always find a time  $t_1 > t_0$  such that

$$\dot{V}(t, z) \leq 0, \quad \forall t_0 \leq t \leq t_1,$$

which implies

$$\int_{t_0}^{t_1} \dot{V}(t, z(t)) dt = V(t_1, z(t_1)) - V(t_0, z(t_0)) \leq 0.$$

Then, from (50), we have

$$\lambda_1 \|z(t_1)\|^2 \leq V(t_1, z(t_1)) \leq V(t_0, z(t_0)) \leq \lambda_2 \|z(t_0)\|^2.$$

Taking

$$\lambda_1 \|z(t_1)\|^2 \leq \lambda_2 \|z(t_0)\|^2,$$

and resolving for  $\|z(t_1)\|$

$$\|z(t_1)\| \leq \delta \|z(t_0)\|;$$

nevertheless, as  $t_1$  can be taken arbitrarily large, we can simply write

$$\|z(t)\| \leq \delta \|z_0\|, \quad \forall t \geq t_0.$$

The previous analysis shows that if  $\|\varphi(t, z)\| \geq \epsilon$  then  $z(t)$  is uniformly bounded and, from (65),  $\dot{V}(t, z) \leq -\lambda_3 \|z(t)\|^2$ . Therefore, for all  $\|\varphi(t, z)\|$  we have

$$\dot{V}(t, z) \leq -[\lambda_3 - \epsilon \zeta_2] \|z\|^2 + \epsilon \zeta_1 \|z\| + \epsilon \zeta_0. \quad (66)$$

Notice that the right hand-side of (66) has the form of (47), provided that inequality (57) is satisfied. Thus, according to Fact 1,  $\dot{V}(t, z) < 0$  for all  $\|z(t)\| \geq \bar{\eta}$  with  $\bar{\eta}$  defined in (59).

Finally, the application of Theorem 1 allows us to conclude that there is a  $T \geq 0$  such that the solution  $z(t)$  of (34) is uniformly ultimately bounded, with uniform bound given by

$$\|z(t)\| \leq \sqrt{\frac{\lambda_2}{\lambda_1}} \max\{\|z(t_0)\|, \bar{\eta}\}, \quad \forall t \geq t_0,$$

and with ultimate bound given by

$$\|z(t)\| \leq \sqrt{\frac{\lambda_2}{\lambda_1}} \bar{\eta}', \quad \forall t \geq t_0 + T, \quad (67)$$

for all  $\bar{\eta}' > \bar{\eta}$ .

△

**Remark 1.** Notice that if  $k_p \rightarrow \infty$  in the PI controller then, from (27),  $\alpha \rightarrow 0$  and, considering (43) and (57), we get  $\lambda_3 = \lambda_{\min}\{Q\} \rightarrow 0$  and  $\epsilon \rightarrow 0$ . Moreover, from (59),  $\epsilon \rightarrow 0$  implies that  $\bar{\eta} \rightarrow 0$ , the uniform bound  $\sigma$  becomes  $\sigma = \sqrt{\frac{\lambda_2}{\lambda_1}} \|z(t_0)\|$  and the system tends to be exponentially stable.

**Remark 2.** It is noteworthy that the joint velocity PI controller

$$\tau = [k_p + \gamma] \bar{v} + [k_i + \gamma \alpha] \xi$$

$$\dot{\xi} = \bar{v}$$

can be also written as

$$\begin{aligned} \tau &= [k_p + \gamma]\bar{v} + \alpha y \\ \frac{1}{k_p + \gamma} \dot{y} &= \bar{v}. \end{aligned}$$

Thus, if  $\gamma \rightarrow \infty$ , then  $\bar{v} \rightarrow 0$  and  $\dot{q} \rightarrow v_d$ . Also, considering (24) and (26),  $\gamma \rightarrow \infty$  implies  $K_p, K_i \rightarrow \infty$ .

From the previous remarks we can conclude that the common assumption of having high-value gains in the inner velocity PI controller of industrial robots leads to small tracking errors during the execution of a motion control task.

### 4.3. Tuning procedure

For design purposes, and according to (22), (24) and (26), we require four scalar parameters ( $k_p, k_i, \alpha$  and  $\gamma$ ) to completely define the inner loop joint velocity PI controller. Additionally, we need to specify the symmetric positive definite gain matrix  $K$  for the outer loop RMRC controller (15).

In this section we propose a simple procedure for the selection of  $k_p, k_i, \alpha, \gamma$  and the smallest and largest eigenvalues of the gain matrix  $K$ , so as to guarantee the uniform ultimate boundedness of the solutions of the overall closed-loop system (34).

It is assumed that the following data are known:

- $k_M, k_C, k_g, \lambda_{\max}\{M\}$  and  $\lambda_{\min}\{M\}$ , which are obtained from the elements of the dynamic model (4), according to (6)–(8).
- $k_J, k_{Jp}$  and  $k_{dJp}$ , the bounds over the jacobian and pseudoinverse jacobian matrices, given by (11), (13) and (14).
- $\|x_d\|_M, \|\dot{x}_d\|_M$  and  $\|\ddot{x}_d\|_M$ , the supreme values over the norms of the desired trajectory and its times derivatives, for all  $t \geq t_0$ , (21).

The tuning procedure is then given as follows:

1. Select an arbitrary value for  $k_p > 0$ .
2. Choose  $k_i$  according to (40), that is

$$k_i \leq \frac{2k_p^2 \lambda_{\min}\{M\}}{\lambda_{\max}^2\{M\} - \lambda_{\min}^2\{M\}}.$$

3. Compute  $\alpha$  as (27):

$$\alpha = \frac{k_i}{k_p}.$$

4. Select the largest and smallest eigenvalues of the gain matrix  $K$  in order to satisfy (46):

$$\lambda_{\max}\{K\} \geq \lambda_{\min}\{K\} > \frac{k_J^2}{4k_p}.$$

5. Find the value of the parameters  $\zeta_0, \zeta_1$  and  $\zeta_2$  in (33), which according to Appendix A, can be computed as

$$\zeta_0 = k_M[k_{dJp}\|\dot{x}_d\|_M + k_{Jp}\|\ddot{x}_d\|_M] + k_C k_{Jp}^2 \|\dot{x}_d\|_M^2 + k_g,$$

$$\begin{aligned} \zeta_1 &= k_M[\lambda_{\max}\{K\}(k_{dJp} + k_{Jp}(k_J + \lambda_{\max}\{K})) + \alpha] \\ &\quad + k_C k_{Jp}[2k_{Jp}\lambda_{\max}\{K\} + \alpha + 1]\|\dot{x}_d\|_M, \\ \zeta_2 &= k_C[k_{Jp}\lambda_{\max}\{K\}(k_{Jp}\lambda_{\max}\{K\} + \alpha + 1) + \alpha]. \end{aligned}$$

6. Compute  $\lambda_1, \lambda_2$  and  $\lambda_3$  according to (38) and (45), that is

$$\begin{aligned} \lambda_1 &= \lambda_{\min}\{P_1\}, \\ \lambda_2 &= \lambda_{\max}\{P_2\}, \\ \lambda_3 &= \lambda_{\min}\{Q\}, \end{aligned}$$

with

$$\begin{aligned} P_1 &= \begin{bmatrix} \frac{1}{2} & 0 & 0 \\ 0 & \alpha k_p + \frac{1}{2}\alpha^2 \lambda_{\min}\{M\} & -\frac{1}{2}\alpha \lambda_{\max}\{M\} \\ 0 & -\frac{1}{2}\alpha \lambda_{\max}\{M\} & \frac{1}{2}\lambda_{\min}\{M\} \end{bmatrix}, \\ P_2 &= \begin{bmatrix} \frac{1}{2} & 0 & 0 \\ 0 & \alpha k_p + \frac{1}{2}\alpha^2 \lambda_{\max}\{M\} & \frac{1}{2}\alpha \lambda_{\max}\{M\} \\ 0 & \frac{1}{2}\alpha \lambda_{\max}\{M\} & \frac{1}{2}\lambda_{\max}\{M\} \end{bmatrix}, \\ Q &= \begin{bmatrix} \lambda_{\min}\{K\} & 0 & -\frac{1}{2}k_J \\ 0 & \alpha k_i & 0 \\ -\frac{1}{2}k_J & 0 & k_p \end{bmatrix}. \end{aligned}$$

7. Select  $\epsilon$  satisfying (57):

$$0 < \epsilon < \frac{\lambda_3}{\zeta_2}.$$

8. Compute the value of  $\bar{\eta}$  using (59):

$$\bar{\eta} = \frac{\epsilon \zeta_1 + \sqrt{\epsilon^2 \zeta_1^2 + 4\epsilon \zeta_0 [\lambda_3 - \epsilon \zeta_2]}}{2[\lambda_3 - \epsilon \zeta_2]}.$$

9. Select  $\sigma$  satisfying (58)

$$\sigma = \sqrt{\frac{\lambda_2}{\lambda_1} \max\{\|z(t_0)\|, \bar{\eta}\}},$$

where we assume that the initial error  $\|z(t_0)\|$  is known.

10. Choose  $\gamma$  in order to satisfy (56), that is,

$$\gamma \geq \frac{1}{\epsilon}[\sigma^2 \zeta_2 + \sigma \zeta_1 + \zeta_0].$$

11. Finally, use (24) and (26) to get the values of the proportional and integral gain matrices  $K_p$  and  $K_i$ , respectively, i.e.

$$\begin{aligned} K_p &= [k_p + \gamma]I, \\ K_i &= [k_i + \alpha\gamma]. \end{aligned}$$

By following this tuning procedure we ensure that all conditions of Proposition 1 are satisfied and

$$\lim_{t \rightarrow \infty} \left\{ \begin{array}{l} \|\tilde{\mathbf{x}}(t)\| \\ \|\tilde{\boldsymbol{\xi}}(t)\| \\ \|\tilde{\mathbf{v}}(t)\| \end{array} \right\} < c,$$

with  $c = \sqrt{\frac{\lambda_2}{\lambda_1}} \bar{\eta}'$  for all  $\bar{\eta}' > \bar{\eta}$ . Thus the control aim (19) is accomplished.

## 5. Conclusions

In this paper, we have dealt with the motion control problem in operational space using the RMRC (kinematic control) plus the intrinsic joint velocity PI controller of the industrial robots, proving uniformly ultimately boundedness of the solutions of the overall closed-loop system.

As far as we know, this is the first time that such a proof is presented, showing that a simple high-gain joint velocity PI control plus a kinematic control scheme is able to ensure the tracking of the operational space desired trajectories in industrial manipulators. Our theoretical contribution integrates previous results to synthesize the proposed stability proof.

As an example of the usefulness of the proposed stability analysis, we present an explicit procedure for obtaining the control gains to guarantee the uniform ultimate boundedness of the tracking errors for robots using this control scheme.

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## Appendix A: Bounding the disturbance term $\mathbf{d}(t, \mathbf{z})$

First, notice that if  $\mathbf{z} = [\tilde{\mathbf{x}}^T \tilde{\boldsymbol{\xi}}^T \tilde{\mathbf{v}}^T]^T$  then

$$\|\tilde{\mathbf{x}}\| \leq \|\mathbf{z}\|, \quad \|\tilde{\boldsymbol{\xi}}\| \leq \|\mathbf{z}\|, \quad \|\tilde{\mathbf{v}}\| \leq \|\mathbf{z}\| \quad (68)$$

### A. 1. Bounds over $\mathbf{v}_d$

The desired joint velocity,  $\mathbf{v}_d$  is given by (15):

$$\mathbf{v}_d = \mathbf{J}(\mathbf{q})^\dagger [\dot{\mathbf{x}}_d + \mathbf{K}\tilde{\mathbf{x}}].$$

From (2), (13) and (21) the bound over  $\mathbf{v}_d$  is given by

$$\begin{aligned} \|\mathbf{v}_d\| &\leq k_{Jp} [\|\dot{\mathbf{x}}_d\|_M + \lambda_{\max}\{\mathbf{K}\} \|\tilde{\mathbf{x}}\|] \\ &\leq k_{Jp} [\|\dot{\mathbf{x}}_d\|_M + \lambda_{\max}\{\mathbf{K}\} \|\mathbf{z}\|], \end{aligned} \quad (69)$$

where we have considered the property (68).

### A. 2. Bounds over $\dot{\mathbf{v}}_d$

From (20),  $\dot{\mathbf{v}}_d$  is given by

$$\dot{\mathbf{v}}_d = \dot{\mathbf{J}}(\mathbf{q})^\dagger [\dot{\mathbf{x}}_d + \mathbf{K}\tilde{\mathbf{x}}] + \mathbf{J}(\mathbf{q})^\dagger [\dot{\mathbf{x}}_d + \mathbf{K}\dot{\tilde{\mathbf{x}}}]$$

and it is bounded as

$$\begin{aligned} \|\dot{\mathbf{v}}_d\| &\leq k_{dJp} [\|\dot{\mathbf{x}}_d\|_M + \lambda_{\max}\{\mathbf{K}\} \|\tilde{\mathbf{x}}\|] \\ &\quad + k_{Jp} [\|\dot{\tilde{\mathbf{x}}}\|_M + \lambda_{\max}\{\mathbf{K}\} \|\dot{\tilde{\mathbf{x}}}\|]. \end{aligned} \quad (70)$$

Using (18) we have

$$\|\dot{\tilde{\mathbf{x}}}\| \leq \lambda_{\max}\{\mathbf{K}\} \|\tilde{\mathbf{x}}\| + k_J \|\tilde{\mathbf{v}}\| \leq [\lambda_{\max}\{\mathbf{K}\} + k_J] \|\mathbf{z}\|,$$

so that (70) becomes

$$\begin{aligned} \|\dot{\mathbf{v}}_d\| &\leq k_{dJp} [\|\dot{\mathbf{x}}_d\|_M + \lambda_{\max}\{\mathbf{K}\} \|\mathbf{z}\|] \\ &\quad + k_{Jp} [\|\dot{\tilde{\mathbf{x}}}\|_M + \lambda_{\max}\{\mathbf{K}\} (\lambda_{\max}\{\mathbf{K}\} + k_J) \|\mathbf{z}\|]. \end{aligned} \quad (71)$$

### A. 3. Bounds over the disturbance term $\mathbf{d}(t, \mathbf{z})$

From (32) we have

$$\mathbf{d}(t, \mathbf{z}) = \mathbf{M}(\mathbf{q}) [\dot{\mathbf{v}}_d + \alpha \tilde{\mathbf{v}}] + \mathbf{C}(\mathbf{q}, \dot{\mathbf{q}}) [\mathbf{v}_d + \alpha \tilde{\boldsymbol{\xi}}] + \mathbf{g}(\mathbf{q}),$$

which is bounded by

$$\begin{aligned} \|\mathbf{d}(t, \mathbf{z})\| &\leq \|\mathbf{M}(\mathbf{q}) [\dot{\mathbf{v}}_d + \alpha \tilde{\mathbf{v}}]\| \\ &\quad + \|\mathbf{C}(\mathbf{q}, \dot{\mathbf{q}}) [\mathbf{v}_d + \alpha \tilde{\boldsymbol{\xi}}]\| + \|\mathbf{g}(\mathbf{q})\|. \end{aligned}$$

We can use (6)–(8) to find an upper bound for each of the terms of the last equation, i.e.,

- Term  $\|M(\mathbf{q})[\dot{\mathbf{v}}_d + \alpha \bar{\mathbf{v}}]\|$ :

$$\begin{aligned}
 & \|M(\mathbf{q})[\dot{\mathbf{v}}_d + \alpha \bar{\mathbf{v}}]\| \\
 & \leq k_M[\|\dot{\mathbf{v}}_d + \alpha \bar{\mathbf{v}}\|, \\
 & \leq k_M[\|\dot{\mathbf{v}}_d\| + \alpha \|\bar{\mathbf{v}}\|], \\
 & \leq k_M[k_{dJp}[\|\dot{\mathbf{x}}_d\|_M + \lambda_{\max}\{\mathbf{K}\}\|\mathbf{z}\|] \\
 & \quad + k_{Jp}[\|\ddot{\mathbf{x}}_d\|_M + \lambda_{\max}\{\mathbf{K}\}(\lambda_{\max}\{\mathbf{K}\} + k_J)\|\mathbf{z}\|] + \alpha \|\bar{\mathbf{v}}\|], \\
 & \leq k_M \underbrace{[k_{dJp}\|\dot{\mathbf{x}}_d\|_M + k_{Jp}\|\ddot{\mathbf{x}}_d\|_M]}_{\beta_1} \\
 & \quad + \underbrace{k_M[\lambda_{\max}\{\mathbf{K}\}(k_{dJp} + k_{Jp}(k_J + \lambda_{\max}\{\mathbf{K}\})) + \alpha]}_{\beta_2} \|\mathbf{z}\|,
 \end{aligned}$$

where we have used (71), and the property (6).

- Term  $\|C(\mathbf{q}, \dot{\mathbf{q}})[\mathbf{v}_d + \alpha \xi]\|$ :

$$\|C(\mathbf{q}, \dot{\mathbf{q}})[\mathbf{v}_d + \alpha \xi]\| \leq k_C \|\dot{\mathbf{q}}\| \|\mathbf{v}_d + \alpha \xi\|,$$

and from (17)

$$\|\dot{\mathbf{q}}\| = \|\mathbf{v}_d - \bar{\mathbf{v}}\| \leq \|\mathbf{v}_d\| + \|\bar{\mathbf{v}}\|,$$

so that

$$\begin{aligned}
 & \|C(\mathbf{q}, \dot{\mathbf{q}})[\mathbf{v}_d + \alpha \xi]\| \leq k_C[\|\mathbf{v}_d\| + \|\bar{\mathbf{v}}\|][\|\mathbf{v}_d\| + \alpha \|\xi\|], \\
 & \leq k_C[\|\mathbf{v}_d\| + \|\mathbf{z}\|][\|\mathbf{v}_d\| + \alpha \|\mathbf{z}\|], \\
 & \leq k_C[\|\mathbf{v}_d\|^2 + (1 + \alpha)\|\mathbf{v}_d\|\|\mathbf{z}\| + \alpha \|\mathbf{z}\|^2],
 \end{aligned}$$

and, by using (69)

$$\begin{aligned}
 & \|C(\mathbf{q}, \dot{\mathbf{q}})[\mathbf{v}_d + \alpha \xi]\| \\
 & \leq k_C [k_{Jp}^2[\|\dot{\mathbf{x}}_d\|_M + \lambda_{\max}\{\mathbf{K}\}\|\mathbf{z}\|]^2 \\
 & \quad + (1 + \alpha)k_{Jp}[\|\dot{\mathbf{x}}_d\|_M + \lambda_{\max}\{\mathbf{K}\}\|\mathbf{z}\|]\|\mathbf{z}\| + \alpha \|\mathbf{z}\|^2], \\
 & \leq \underbrace{k_C k_{Jp}^2}_{\beta_3} \|\dot{\mathbf{x}}_d\|_M^2 + \underbrace{k_C k_{Jp}[2k_{Jp}\lambda_{\max}\{\mathbf{K}\} + \alpha + 1]}_{\beta_4} \|\dot{\mathbf{x}}_d\|_M \|\mathbf{z}\| \\
 & \quad + \underbrace{k_C[k_{Jp}\lambda_{\max}\{\mathbf{K}\}(k_{Jp}\lambda_{\max}\{\mathbf{K}\} + \alpha + 1) + \alpha]}_{\beta_5} \|\mathbf{z}\|^2.
 \end{aligned}$$

- Term  $\|\mathbf{g}(\mathbf{q})\|$ : Using (8), it is simply

$$\|\mathbf{g}(\mathbf{q})\| \leq k_g = \beta_6.$$

Therefore, the bound over the disturbance term (32) is given by

$$\begin{aligned}
 & \|\mathbf{d}(t, \mathbf{z})\| \leq \beta_1 + \beta_2 \|\mathbf{z}\| + \beta_3 + \beta_4 \|\mathbf{z}\| + \beta_5 \|\mathbf{z}\|^2 + \beta_6, \\
 & \leq \underbrace{[\beta_1 + \beta_3 + \beta_6]}_{\zeta_0} + \underbrace{[\beta_2 + \beta_4]}_{\zeta_1} \|\mathbf{z}\| + \underbrace{\beta_5}_{\zeta_2} \|\mathbf{z}\|^2,
 \end{aligned} \tag{72}$$

where  $\zeta_0$ ,  $\zeta_1$  and  $\zeta_2$  are strictly positive constants.

## Appendix B: Proof of theorem 1

**Proof:** The proof of the Theorem 1 is given by contradiction, following the steps in ref. [11].

Let us define

$$\bar{r} = \max\{\|\mathbf{z}(t_0)\|, \eta\}. \tag{73}$$

A. From (73), we have that

$$\|\mathbf{z}(t_0)\| \leq \bar{r}, \text{ and } \eta \leq \bar{r}. \tag{74}$$

Also, by (50),  $\lambda_1 \leq \lambda_2$  so that  $\bar{r} \leq \sqrt{\frac{\lambda_2}{\lambda_1}} \bar{r}$ . Thus

$$\|\mathbf{z}(t_0)\| \leq \bar{r} \leq \sqrt{\frac{\lambda_2}{\lambda_1}} \bar{r}.$$

Now suppose there is a time  $t_2$ , with  $t_0 < t_2$ , such that

$$\|\mathbf{z}(t_2)\| > \sqrt{\frac{\lambda_2}{\lambda_1}} \bar{r} \tag{75}$$

in order that

$$\|\mathbf{z}(t_0)\| \leq \bar{r} \leq \sqrt{\frac{\lambda_2}{\lambda_1}} \bar{r} < \|\mathbf{z}(t_2)\|.$$

By continuity of  $\mathbf{z}(t)$ , there is a  $t_1$ , with  $t_0 \leq t_1 < t_2$ , such that

$$\|\mathbf{z}(t_1)\| = \bar{r} \tag{76}$$

and  $\|\mathbf{z}(t)\| \geq \bar{r}$ ,  $\forall t_1 \leq t \leq t_2$ , which considering (74), results in

$$\|\mathbf{z}(t)\| \geq \eta, \forall t_1 \leq t \leq t_2. \tag{77}$$

From (50)–(51), taking into account (77) and (76) we have

$$\begin{aligned}
 & \lambda_1 \|\mathbf{z}(t_2)\|^2 \leq V(t_2, \mathbf{z}(t_2)) \\
 & = V(t_1, \mathbf{z}(t_1)) + \int_{t_1}^{t_2} \dot{V}(\tau, \mathbf{z}(\tau)) d\tau \\
 & \leq \lambda_2 \|\mathbf{z}(t_1)\|^2 + \int_{t_1}^{t_2} (0) d\tau \\
 & = \lambda_2 \bar{r}^2,
 \end{aligned}$$

and it can be implied that

$$\|\mathbf{z}(t_2)\| \leq \sqrt{\frac{\lambda_2}{\lambda_1}} \bar{r}$$

but this contradicts supposition (75), hence

$$\|\mathbf{z}(t)\| \leq \sqrt{\frac{\lambda_2}{\lambda_1}} \bar{r}, \forall t \geq t_0.$$

B. If  $\|z(t_0)\| \leq \eta'$  then, in view of the uniform boundedness result in A, we have

$$\|z(t)\| \leq \sqrt{\frac{\lambda_2}{\lambda_1}} \bar{r} \leq \sqrt{\frac{\lambda_2}{\lambda_1}} \eta', \quad \forall t \geq t_0,$$

so that  $T = 0$ .

Next, consider  $\|z(t_0)\| > \eta'$  and suppose that

$$\|z(t)\| > \eta' > \eta, \quad \forall t \geq t_0 \quad (78)$$

then, in Fact 1 we have that  $g(\|z(t)\|)$  is a strictly decreasing function, so that

$$g(\|z(t)\|) < g(\eta') < 0, \quad \forall t \geq t_0. \quad (79)$$

In view of (78), we can use (50), (51) and (79) to get

$$\begin{aligned} \lambda_1 \|z(t)\|^2 &\leq V(t, z(t)) \\ &= V(t_0, z(t_0)) + \int_{t_0}^t \dot{V}(\tau, z(\tau)) d\tau \\ &\leq \lambda_2 \|z(t_0)\|^2 + \int_{t_0}^t g(\eta') d\tau \end{aligned}$$

which implies that

$$\lambda_1 \|z(t)\|^2 \leq \lambda_2 \|z(t_0)\|^2 + g(\eta')[t - t_0], \quad \forall t \geq t_0. \quad (80)$$

Notice, however, that for each  $\eta'$  there is a time  $T > 0$ , given by

$$T = \frac{\lambda_1 \eta'^2 - \lambda_2 \|z(t_0)\|^2}{g(\eta')}$$

such that  $t = t_0 + T$ , then (80) becomes

$$\lambda_1 \|z(t)\|^2 \leq \lambda_1 \eta'^2 \quad \forall t \geq t_0 + T$$

so that

$$\|z(t)\| \leq \eta' \quad \forall t \geq t_0 + T.$$

But this contradicts supposition (78). Hence, there must be a  $t_1$ , with  $t_0 \leq t_1 \leq t_0 + T$  such that  $\|z(t_1)\| = \eta'$  for some  $t \geq t_1$ . Then, as a consequence of the uniform boundedness result A,

$$\|z(t)\| \leq \sqrt{\frac{\lambda_2}{\lambda_1}} \eta', \quad \forall t \geq t_1,$$

and

$$\|z(t)\| \leq \sqrt{\frac{\lambda_2}{\lambda_1}} \eta', \quad \forall t \geq t_0 + T. \quad \triangle$$