# DYNAMIC POSITIONING CONTROL OF SHIPS VIA RELAY OBSERVER DESIGN

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# DYNAMIC POSITIONING CONTROL OF SHIPS VIA RELAY OBSERVER DESIGN

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## ABSTRACT

A dynamically positioned ship maintains its position (fixed location or predetermined track) exclusively by means of active thrusters. In this paper, a control scheme for dynamic positioning control of ships using a relay observer design is presented. Our analysis relies on nonsmooth strict Lyapunov functions to demonstrate global asymptotic stability of the closed-loop system equilibrium. Using simulation results we illustrate the good performance of the proposed scheme under noisy ship position measurements.


***KeyWords:*** Ships, positioning control, relay observers, asymptotic stability.

## I. INTRODUCTION

A ship control system where the ship position is controlled exclusively by means of thrusters and main propellers aft of the ship defines a dynamic positioning system. Different type of thrusters devices are available like tunnel thrusters which can produce thrust in the sideway direction and azimuth thrusters which are usually mounted under the hull of the ship [1,2].

The observer design for implementation of controllers applied to ships is important because the instruments used to navigate the ship, in many cases, only give position information. Thus, controller design for ships requires estimation of the velocities and filtering of the position measurements. In conventional feedback ship control systems, the ship velocities are often estimated by using Kalman filtering, under the assumption that the ship model is linear around the operation points. The provisional nature of the linearization approach does not guarantee the desired stability convergence properties, which mean a poor performance of the control/observer system. Considering a nonlinear ship model, the accuracy of the observation of state vector can be improved. In this respect, most of the observers obtained for ships (see, *e.g.*, [3,4], and [2]) involves asymptotic or exponential convergence of the observation to its true values. However, using variable structure theory, it is also possible to design an observer with good performance. Moreover, the discontinuous being of the relay observers improves the robustness properties of the control system against external perturbations [5,6].

We use these ideas to design a controller based on a relay observer to solve the problem of dynamic positioning of ships. Based in previous studies on dynamic positioning control of ships [1-4,7,8], the design of controllers with relay observers for ships is a novel result. The present work results in an application of the theory of variable structure systems, like to the surveyed in [9] and [10], in the problem of dynamic positioning control of ships. We use analysis tools for right-hand discontinuous systems. Namely, nonsmooth strict Lyapunov functions, which allow us to conclude global asymptotic stability of the system equilibrium without invoking the LaSalle theorem, that in general, for dynamic systems governed by differential inclusions fails because of the ambiguous behavior of the system solutions [5].

The organization of the paper is as follows. In Section 2 the problem statement of position control of ships is presented. Section 3 deals with the relay observer design, which contains switching terms via sign function. Section 4 illustrates a new control law for dynamic ship positioning using the designed relay observer introduced in Section 3. Section 5 is dedicated to present simulation results to support our theoretical results. Finally, in Sec-

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tion 6, some concluding remarks are written.

Throughout this paper the following notation will be adopted. $\lambda_{min}\{A\}$ and $\lambda_{max}\{A\}$ denote the minimum and maximum eigenvalues of a symmetric positive definite matrix $A \in R^{n\times n}$, respectively. $\|x\| = \sqrt{x^T x}$ stands for the Euclidean norm of vector $x \in R^n$, $\|x\|_1 = |x_1| + \ldots + |x_n|$ stands for the sum norm (or $l_1$ norm) on $R^n$, and $\|B\| = \sqrt{\lambda_{max}\{B^T B\}}$ is the induced norm of a matrix $B \in R^{n\times n}$. The abbreviation *a.e.* means "almost everywhere with respect to the Lebesgue measure".

In this paper we used properties of the tangent hyperbolic function, which is defined as

$$\tanh(x) = \frac{e^x - e^{-x}}{e^x + e^{-x}}.$$

The tangent hyperbolic function can be arranged in a vector in the following way:

$$\tanh(z) = [\tanh(z_1), \ \ldots, \ \tanh(z_n)]^T,$$

and the following properties are accomplished by $\tanh(z)$

(a) The Euclidean norm of $\tanh(z)$ satisfies:

$$\|\tanh(z)\| \leq \begin{cases} \|z\| & \forall z \in R^n. \\ \sqrt{n} & \forall z \in R^n. \end{cases}$$

(b) The time derivative of $\tanh(z)$ is given by

$$\frac{d}{dt}\tanh(z) = \mathrm{Sech}^2(z)\,\dot{z},$$

where $\mathrm{Sech}^2(z) = \mathrm{diag}\{\mathrm{sech}^2(z_1), \ldots, \mathrm{sech}^2(z_n)\}$ and

$$\mathrm{Sech}(x) = \frac{2}{e^x + e^{-x}} = \frac{1}{\cosh(x)}.$$

(c) The maximum eigenvalue of the matrix $\mathrm{Sech}^2(z)$ is one for all $z \in R^n$; *i.e.*,

$$\lambda_{max}\{\mathrm{Sech}^2(z)\} = 1 \qquad \forall z \in R^n.$$

## II. PROBLEM FORMULATION

We consider that the ship dynamic model is given by the next Lagrangian expression [4,1]:

$$M\dot{\nu} + D\nu + K\eta = \tau, \tag{1}$$

$$\dot{\eta} = J(\eta)\,\nu, \tag{2}$$

$$J(\eta) = \begin{bmatrix} \cos(\psi) & -\sin(\psi) & 0 \\ \sin(\psi) & \cos(\psi) & 0 \\ 0 & 0 & 1 \end{bmatrix}, \tag{3}$$

where $M$ is the inertia matrix which include hydrodynamic inertia, $D$ is the damping matrix, $K$ is the anchor forces and moments matrix, $J(\eta)$ is the rotational matrix in yaw angle $\psi$, and $\tau$ is the vector control forces. The above dynamics describe the ship motion with respect to the earth-fixed positions $(x, y)$ and yaw angle $\psi$ of the vessel expressed in vector form $\eta = [x, y, \psi]^T$, whereas the body-fixed velocities are represented by the vector $\nu = [u, v, r]^T$. The elements in these state vectors describes the surge, sway, and yaw modes, respectively [4]; see Figs. 1 and 2.

We used the following assumptions:

(H1) The inertia matrix $M$ is constant positive definite and symmetric; *i.e.*, $M = M^T > 0$.

(H2) The damping matrix $D$ is constant such that

$$x^T D\, x = \frac{1}{2} x^T [D + D^T]\, x > 0 \qquad \forall x \neq 0.$$

(H3) The time derivative of the matrix $J(\eta)$ satisfies the following bound

$$\left\| \frac{d}{dt} J(\eta) \right\| \leq \|\nu\| \qquad \forall \nu \in R^3. \tag{4}$$

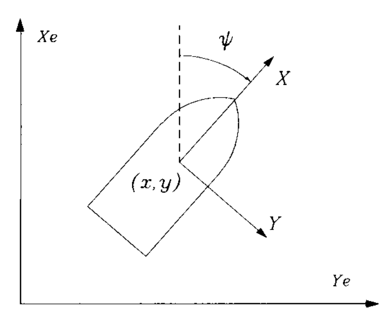


Fig. 1. Earth-fixed and ship coordinates frames.

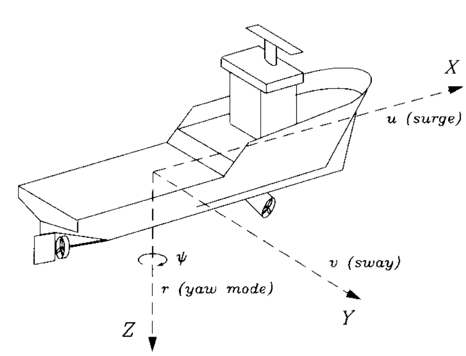


Fig. 2. Definitions of variables in surge ($u$), sway ($v$), and yaw ($r$) modes for a marine vessel.

The assumption (*H1*) is a common assumption for most low-speed applications [4,2]. Assumption (*H2*) is a property of the Lagrangian model in (1); *i.e.*, the matrix $D$ is strictly positive. Using the relation (2) and the definition of $J(\eta)$ in (3), it is possible to demonstrate that assumption (*H3*) always holds for all $\eta, v \in R^3$.

The state-space representation of the ship dynamic model is as follows:

$$\dot{\eta} = J(\eta)\,\nu\,, \tag{5}$$

$$\dot{\nu} = A_1\eta + A_2\nu + B\tau\,, \tag{6}$$

where

$$A_1 = -M^{-1}K, \quad A_2 = -M^{-1}D, \quad B = M^{-1}\,.$$

In this paper the problem of dynamic positioning of ships using only attitude measurements is considered. This problem results in many situations where the ship velocity is not available from sensors. Thus, the yaw angle is measured using a gyro compass, and the positions $x$ and $y$ are assumed to be measured employing a global position system (GPS), for example, the differential GPS NAVSTAR [4]. Therefore, under the assumptions (*H1*), (*H2*), and (*H3*), the problem is, given the desired constant position vector $\eta_d$, design a control law such that

$$\lim_{t\to\infty}\eta(t) = \eta_d \qquad \text{and} \qquad \lim_{t\to\infty}\nu(t) = 0\,,$$

via nonlinear observer; *i.e.*, given a dynamic of the form

$$\dot{\hat{\eta}} = \Phi_1(\eta, \hat{\eta}, \tilde{\eta}, \hat{\nu}), \quad \hat{\eta} \in R^3\,,$$

$$\dot{\hat{\nu}} = \Phi_2(\eta, \hat{\eta}, \tilde{\eta}, \hat{\nu}), \quad \hat{\nu} \in R^3\,,$$

where

$$\tilde{\eta} = \eta - \hat{\eta}\,, \tag{7}$$

is the position estimation error, $\Phi_1(\eta, \hat{\eta}, \tilde{\eta}, \hat{\nu})$ and $\Phi_2(\eta, \hat{\eta}, \tilde{\eta}, \hat{\nu})$ are nonlinear functions, the following limits are satisfied

$$\lim_{t\to\infty}\tilde{\eta} = 0 \qquad \lim_{t\to\infty}\tilde{\nu} = 0\,,$$

where

$$\tilde{\nu} = \nu - \hat{\nu}\,, \tag{8}$$

is the velocity estimation error.

## III. RELAY OBSERVER DESIGN

In this section we present a modification of the observer design given by [4], which is as follows:

$$\dot{\hat{\eta}} = J(\eta)\hat{\nu} + K_1\tilde{\eta}\,, \tag{9}$$

$$\dot{\hat{\nu}} = A_1\hat{\eta} + A_2\hat{\nu} + B\tau + K_2(\eta)\tilde{\eta}\,, \tag{10}$$

where the gains $K_1$ and $K_2(\eta)$ are designed in such way that the next equations are satisfied:

$$J(\eta)^T P_1 + P_2[A_1 - K_2(\eta)] = 0\,, \tag{11}$$

$$K_1^T P_1 = Q_1, \quad Q_1 = Q_1^T, \quad Q_1 > 0\,, \tag{12}$$

$$\frac{1}{2}[P_2A_2 + A_2^TP_2] = -Q_2, \quad Q_2 = Q_2^T, \quad Q_2 > 0\,, \tag{13}$$

where $P_1$ and $P_2$ are two given symmetric positive definite matrices. At this stage, it is important to remark that Eq. (13) implies that matrix $A_2$ is Hurwitz (this assumption is relaxed in [11], where a method is proposed to find proper observer gains $K_1$ and $K_2(\eta)$ when $A_2$ is not Hurwitz).

Based on (9)-(10), we introduce the following relay observer

$$\dot{\hat{\eta}} = J(\eta)\hat{\nu} + K_1\tilde{\eta} + K_{\alpha 1}\,\mathrm{sgn}(\tilde{\eta})\,, \tag{14}$$

$$\dot{\hat{\nu}} = A_1\hat{\eta} + A_2\hat{\nu} + B\tau + K_2(\eta)\tilde{\eta} + K_{\alpha 2}(\eta)\,\mathrm{sgn}(\tilde{\eta})\,, \tag{15}$$

where $K_{\alpha 1} = \mathrm{diag}\{k_{\alpha 11}, k_{\alpha 12}, k_{\alpha 13}\}$, $K_{\alpha 1i} > 0$, $i = 1, 2, 3$, and $K_{\alpha 2}(\eta)$ is such that

$$J(\eta)^T - P_2K_{\alpha 2}(\eta) = 0\,. \tag{16}$$

The values of $K_1$ and $K_2(\eta)$ are obtained according to (11)-(13). Note that

$$\mathrm{sgn}(\tilde{\eta}) = \begin{bmatrix} \mathrm{sgn}(x - \hat{x}) \\ \mathrm{sgn}(y - \hat{y}) \\ \mathrm{sgn}(\psi - \hat{\psi}) \end{bmatrix},$$

where the sign function is defined as

$$\mathrm{sgn}(x) = \begin{cases} 1 & \text{for } x > 0\,, \\ [-1,\ 1] & \text{for } x = 0\,, \\ -1 & \text{for } x < 0\,. \end{cases}$$

Utilizing the state space expression of the ship dynamics (5)-(6), the presented relay observer (14)-(15), and from equations (7)-(8), the observation error dynamics can be written as

$$\dot{\tilde{\eta}} = J(\eta)\tilde{\nu} - K_1\tilde{\eta} - K_{\alpha 1}\,\mathrm{sgn}(\tilde{\eta})\,, \tag{17}$$

$$\dot{\tilde{\nu}} = [A_1 - K_2]\tilde{\eta} + A_2\tilde{\nu} - K_{\alpha 2}(\eta)\,\mathrm{sgn}(\tilde{\eta})\,. \tag{18}$$

Now, let us introduce the nonsmooth Lyapunov function

$$V_o(\tilde{\eta},\tilde{\nu}) = \frac{1}{2}(\tilde{\eta}^T P_1 \tilde{\eta} + \tilde{\nu}^T P_2 \tilde{\nu}) + \| \tilde{\eta} \|_1 ,$$

which is positive definite and proper by construction. The time derivative of $V_o(\tilde{\eta},\tilde{\nu})$ along of the trajectories of the error dynamics (17)-(18) produces

$$\begin{aligned}
\dot{V}_o(\tilde{\eta},\tilde{\nu}) &= \tilde{\eta}^T P_1 [J(\eta)\tilde{\nu} - K_1\tilde{\eta} - K_{\alpha 1}\,\mathrm{sgn}(\tilde{\eta})] \\
&\quad + \tilde{\nu}^T P_2 [(A_1 - K_2)\tilde{\eta} + A_2\tilde{\nu} - K_{\alpha 2}\,\mathrm{sgn}(\tilde{\eta})] \\
&\quad + \mathrm{sgn}^T(\tilde{\eta})\,[J(\eta)\tilde{\nu} - K_1\tilde{\eta} - K_{\alpha 1}\,\mathrm{sgn}(\tilde{\eta})] \\
&\le -\tilde{\eta}^T P_1 K_1 \tilde{\eta} + \frac{1}{2}\tilde{\nu}^T (P_2 A_2 + A_2^T P_2)\tilde{\nu} \\
&\quad + \tilde{\nu}^T [J^T(y) P_1 + P_2(A_1 - K_2)]\,\tilde{\eta} \\
&\quad + \tilde{\nu}^T [J^T(y) - P_2 K_{\alpha 2}(\eta)]\,\mathrm{sgn}(\tilde{\eta}) - k_m \qquad a.e.,
\end{aligned}$$

where $k_m = \min\{k_{\alpha 1 i}\}$. Invoking Eqs. (11)-(13) and (16), we obtain

$$\dot{V}_o(\tilde{\eta},\tilde{\nu}) \le -\tilde{\eta}^T Q_1 \tilde{\eta} - \tilde{\nu}^T Q_2 \tilde{\nu} \qquad a.e. \tag{19}$$

We have shown that $\dot{V}_o(\tilde{\eta},\tilde{\nu})$ is negative definite beyond the manifold $\tilde{\eta} = 0$. Thus, every trajectory started outside of the discontinuity manifold ($\tilde{\eta} = 0$) converges to it asymptotically. So, it remains to prove that trajectories originated into this discontinuity manifold are asymptotically stable. To this end, we have that $\dot{\tilde{\eta}} = 0$ at the switching manifold $\tilde{\eta} = 0$, and from (17) we have

$$\mathrm{sgn}(\tilde{\eta}) := K_{\alpha 1}^{-1} J(\eta)\,\tilde{\nu} . \tag{20}$$

Using Eq. (20) into the dynamics of $\tilde{\nu}$ in (18) (equivalent control method [9]), we have

$$\dot{\tilde{\nu}} = A_2\tilde{\nu} - K_{\alpha 2}(\eta) K_{\alpha 1}^{-1} J(\eta)\,\tilde{\nu} ,$$

which can be rewritten as

$$\dot{\tilde{\nu}} = [A_2 - P_2^{-1} J(\eta)^T K_{\alpha 1}^{-1} J(\eta)]\,\tilde{\nu} , \tag{21}$$

after using Eq. (16). Thus, taking the time derivative of $V_o(0,\tilde{\nu}) = \frac{1}{2}\tilde{\nu}^T P_2 \tilde{\nu}$, and employing (13), we obtain

$$\begin{aligned}
\dot{V}_o(0,\tilde{\nu}) &= \tilde{\nu}^T P_2 A_2 \tilde{\nu} - \tilde{\nu}^T J(\eta)^T K_{\alpha 1} J(\eta)\,\tilde{\nu} \\
&= -\tilde{\nu}^T Q_1 \tilde{\nu} - \tilde{\nu}^T J(\eta)^T K_{\alpha 1} J(\eta)^T \tilde{\nu} .
\end{aligned}$$

Since $V_o(\tilde{\eta},\tilde{\nu})$ is positive definite and radially unbounded, and its time derivative $\dot{V}_o(\tilde{\eta},\tilde{\nu})$ is negative definite into and out of the discontinuity manifold, we conclude that the solutions $[\tilde{\eta}(t)^T\ \tilde{\nu}(t)^T]^T$ of the error system (17)-(18) converge asymptotically to the trivial equilibrium point for all initial conditions $[\tilde{\eta}(0)^T\ \tilde{\nu}(0)^T]^T$.

## IV. DYNAMIC POSITION CONTROL VIA RELAY OBSERVER

To solve the position control problem formulated in Section 2 via relay observer (14) and (15), inspired in [2], the next control law is proposed:

$$\tau = -[K_\nu - D]\,\hat{\nu} - J(\eta)^T K_p e_\eta + K_\eta \tilde{\eta} + K\eta , \tag{22}$$

where

$$e_\eta = \eta - \eta_d , \tag{23}$$

which denotes the position error, $K_\nu$ is a symmetric positive definite matrix, and $K_\eta$ is a constant matrix.

Substituting the control law (22) into the ship model (1), and using Eq. (23), the closed-loop system yields

$$\dot{e}_\eta = -J(e_\eta + \eta_d)\,\nu , \tag{24}$$

$$\begin{aligned}
\dot{\nu} &= -M^{-1} K_\nu \nu - M^{-1} J(\eta)^T K_p\, e_\eta \\
&\quad + M^{-1}[K_\nu - D]\,\tilde{\nu} + M^{-1} K_\eta\, \tilde{\eta} .
\end{aligned} \tag{25}$$

Thus, the overall closed-loop system is given by Eqs. (17)-(18) and (24)-(25).

To prove asymptotic stability of the trivial equilibrium point, consider the next nonsmooth strict Lyapunov function

$$\begin{aligned}
V(e_\eta,\nu,\tilde{\eta},\tilde{\nu}) &= \frac{1}{2}\tilde{\eta}^T P_1\, \tilde{\eta} + \frac{1}{2}\tilde{\nu}^T P_2\, \tilde{\nu} + \| \tilde{\eta} \|_1 + \frac{\epsilon}{2}\nu^T M \nu \\
&\quad + \frac{\epsilon}{2} e_\eta^T K_p e_\eta + \alpha \tanh(e_\eta)^T K_p J(\eta)\, M\nu ,
\end{aligned}$$

where $\epsilon$ and $\alpha$ are strictly positive constants, and matrices $P_1$ and $P_2$ are positive definite matrices involved in Eqs. (11)-(13). The Lyapunov function $V(e_\eta,\nu,\tilde{\eta},\tilde{\nu})$ is positive definite provided that $\epsilon$ and $\alpha$ satisfy the following relationship

$$\alpha < \epsilon \sqrt{\frac{\lambda_{min}\{K_p\}\,\lambda_{min}\{M\}}{\lambda_{max}\{K_p\}\,\lambda_{max}\{M\}}} . \tag{26}$$

The time derivative of $V(e_\eta,\nu,\tilde{\eta},\tilde{\nu})$ along of the overall closed-loop system trajectories (17)-(18), and

(24)-(25), yields

$$\dot{V}(e_\eta, \nu, \tilde{\eta}, \tilde{\nu}) = \tilde{\eta}^T P_1[J(\eta)\tilde{\nu} - K_1\tilde{\eta} - K_{\alpha 1}\,\mathrm{sgn}(\tilde{\eta})]$$
$$+\tilde{\nu}^T P_2[(A_1 - K_2)\tilde{\eta} + A_2\tilde{\nu} - K_{\alpha 2}(\eta)\,\mathrm{sgn}(\tilde{\eta})]$$
$$+ \mathrm{sgn}(\tilde{\eta})^T[J(\eta)\tilde{\nu} - K_1\tilde{\nu} - K_{\alpha 1}\,\mathrm{sgn}(\tilde{\eta})]$$
$$-\epsilon\, \nu^T K_\nu \nu + \epsilon\, \nu^T[K_\nu - D]\,\tilde{\nu} + \epsilon\, \nu^T K_\eta \tilde{\eta}$$
$$-\alpha \tanh(e_\eta)^T K_p J(\eta) K_\nu \nu - \alpha \tanh(e_\eta)^T K_p J(\eta) J(\eta)^T K_p e_\eta$$
$$+\alpha \tanh(e_\eta)^T K_p J(\eta)[K_\nu - D]\,\tilde{\nu} + \alpha \tanh(e_\eta)^T K_p J(\eta)\, K_\eta \tilde{\eta}$$
$$+\alpha \tanh(e_\eta)^T K_p \dot{J}(\eta)\, M\nu$$
$$-\alpha \nu^T M\, J(\eta)^T K_p\, \mathrm{Sech}^2(e_\eta) J(\eta)\nu \qquad a.e.$$

By virtue of Eqs. (11)-(13), and (16), and after some algebra manipulations, the following upper bound on $\dot{V}(e_\eta, \nu, \tilde{\eta}, \tilde{\nu})$ is obtained:

$$\dot{V}(\eta, \nu, \tilde{\eta}, \tilde{\nu}) \le -\tilde{\eta}^T Q_1 \tilde{\eta} - \tilde{\nu}^T Q_2 \tilde{\nu}$$
$$- \epsilon\, \lambda_{min}\{K_\nu\} \| \nu \|^2 + \epsilon\, \bar{k}_1 \| \nu \| \| \tilde{\nu} \| + \epsilon\, \bar{k}_2 \| \nu \| \| \tilde{\eta} \|$$
$$+ \alpha\, \lambda_{max}\{K_p\}\, \lambda_{max}\{K_\nu\} \| \tanh(e_\eta) \| \| \nu \|$$
$$- \alpha\, \lambda_{min}\{Q_{e_\eta}\} \| \tanh(e_\eta) \|^2$$
$$+ \alpha\, \bar{k}_1\, \lambda_{max}\{K_p\} \| \tanh(e_\eta) \| \| \tilde{\nu} \|$$
$$+ \alpha\, \bar{k}_2 \lambda_{max}\{K_p\} \| \tanh(e_\eta) \| \| \tilde{\eta} \| + \alpha\, \gamma_1 \| \nu \|^2,$$

where

$$\bar{k}_1 = \| K_\nu - D \|, \qquad \bar{k}_2 = \| K_\eta \|,$$
$$Q_{e_\eta} = K_p J(\eta) J(\eta)^T K_p\,,$$

and

$$\gamma_1 = \sqrt{n}\, \lambda_{max}\{K_p\}\, \lambda_{max}\{M\} + \lambda_{max}\{K_p\}\, \lambda_{max}\{M\}\,.$$

Let us notice that the fact $\|J(\eta)\| = 1$ for all $\eta \in R^3$, properties of the tangent hyperbolic function, and property (*H3*) in (4) were used.

To establish a conclusion on the convergence of the state space solutions of the overall closed-loop system, we construct a matrix form of the upper bound on $\dot{V}(e_\eta, \nu, \tilde{\eta}, \tilde{\nu})$ as follows:

$$\dot{V}(e_\eta, \nu, \tilde{\eta}, \tilde{\nu}) \le -\begin{bmatrix} \|\tilde{\nu}\| \\ \|\tilde{\eta}\| \\ \|\nu\| \end{bmatrix}^T \bar{Q}_1 \begin{bmatrix} \|\tilde{\nu}\| \\ \|\tilde{\eta}\| \\ \|\nu\| \end{bmatrix}$$
$$-\begin{bmatrix} \|\tilde{\eta}\| \\ \|\nu\| \end{bmatrix}^T \bar{Q}_2 \begin{bmatrix} \|\tilde{\eta}\| \\ \|\nu\| \end{bmatrix}$$
$$-\begin{bmatrix} \|\tilde{\eta}\| \\ \|\tanh(e_\eta)\| \end{bmatrix}^T \bar{Q}_3 \begin{bmatrix} \|\tilde{\eta}\| \\ \|\tanh(e_\eta)\| \end{bmatrix}$$
$$-\begin{bmatrix} \|\nu\| \\ \|\tanh(e_\eta)\| \end{bmatrix}^T \bar{Q}_4 \begin{bmatrix} \|\nu\| \\ \|\tanh(e_\eta)\| \end{bmatrix}$$
$$-\begin{bmatrix} \|\tilde{\nu}\| \\ \|\tanh(e_\eta)\| \end{bmatrix}^T \bar{Q}_5 \begin{bmatrix} \|\tilde{\nu}\| \\ \|\tanh(e_\eta)\| \end{bmatrix} \quad a.e., \qquad (27)$$

where

$$\bar{Q}_1 = \begin{bmatrix} \frac{1}{2}\lambda_{min}\{Q_2\} & 0 & -\frac{\epsilon}{2}\bar{k}_1 \\ 0 & \frac{1}{3}\lambda_{min}\{Q_1\} & -\frac{\epsilon}{2}\bar{k}_2 \\ -\frac{\epsilon}{2}\bar{k}_1 & -\frac{\epsilon}{2}\bar{k}_2 & \frac{1}{3}[\epsilon\, \lambda_{min}\{K_\nu\} - \alpha\gamma_1] \end{bmatrix},$$

$$\bar{Q}_2 = \begin{bmatrix} \frac{1}{3}\lambda_{min}\{Q_1\} & 0 \\ 0 & \frac{1}{3}[\epsilon\, \lambda_{min}\{K_\nu\} - \alpha\gamma_1] \end{bmatrix},$$

$$\bar{Q}_3 = \begin{bmatrix} \frac{1}{3}\lambda_{min}\{Q_1\} & -\frac{1}{2}\alpha\lambda_{max}\{K_p\}\,\bar{k}_2 \\ -\frac{1}{2}\alpha\lambda_{max}\{K_p\}\,\bar{k}_2 & \frac{1}{3}\alpha\lambda_{min}\{Q_{e_\eta}\} \end{bmatrix},$$

$$\bar{Q}_4 = \begin{bmatrix} \frac{1}{3}[\epsilon\, \lambda_{min}\{K_\nu\} - \alpha\gamma_1] & -\frac{1}{2}\alpha\lambda_{max}\{K_p\}\lambda_{max}\{K_\nu\} \\ -\frac{1}{2}\alpha\lambda_{max}\{K_p\}\lambda_{max}\{K_\nu\} & \frac{1}{3}\alpha\lambda_{min}\{Q_{e_\eta}\} \end{bmatrix},$$

$$\bar{Q}_5 = \begin{bmatrix} \frac{1}{2}\lambda_{min}\{Q_2\} & -\frac{1}{2}\alpha\, \bar{k}_1\, \lambda_{max}\{K_p\} \\ -\frac{1}{2}\alpha\, \bar{k}_1\, \lambda_{max}\{K_p\} & \frac{1}{3}\alpha\, \lambda_{min}\{Q_{e_\eta}\} \end{bmatrix}.$$

To demonstrate that $\dot{V}(e_\eta, \nu, \tilde{\eta}, \tilde{\nu})$ is negative definite *a.e.*, *i.e.*, it is a negative definite function beyond the switching manifold $\tilde{\eta}=0$, it is necessary to find sufficient conditions for matrices $\overline{Q}_1$, $\overline{Q}_2$, $\overline{Q}_3$, $\overline{Q}_4$, and $\overline{Q}_5$, to be positive definites. The sufficient conditions for $\overline{Q}_1$ to be positive definite are that

$$\alpha < \frac{\epsilon\,\lambda_{min}\{K_v\}}{\gamma_1}, \tag{28}$$

and

$$\alpha < \frac{\epsilon\,[4\lambda_{min}\{Q_1\}\lambda_{min}\{K_v\} - \epsilon\, 9\overline{k}_2^2]}{4\lambda_{min}\{Q_1\}\gamma_1}, \tag{29}$$

where (29) implies that

$$\epsilon < \frac{4\lambda_{min}\{Q_1\}\lambda_{min}\{K_v\}}{9\overline{k}_2}, \tag{30}$$

must be achieved, because $\alpha$ and $\epsilon$ are strictly positive constants. The matrix $\overline{Q}_2$ is positive definite under inequality (28), while matrix $\overline{Q}_3$ is positive definite if

$$\alpha < \frac{4\lambda_{min}\{Q_1\}\lambda_{min}\{Q_{e_\eta}\}}{9\lambda_{max}\{K_p\}^2\overline{k}_2^2}. \tag{31}$$

The conditions for positive definiteness of $\overline{Q}_4$ are inequality (28) and

$$\alpha < \frac{\epsilon\,\lambda_{min}\{K_v\}\lambda_{min}\{Q_{e_\eta}\}}{\gamma_1\lambda_{min}\{Q_{e_\eta}\} + \frac{9}{4}\lambda_{max}\{K_p\}^2\lambda_{max}\{K_v\}^2}. \tag{32}$$

Finally, the inequality

$$\alpha < \frac{2\lambda_{min}\{Q_2\}\lambda_{min}\{Q_{e_\eta}\}}{3\lambda_{max}\{K_p\}^2\overline{k}_1^2}. \tag{33}$$

must hold for $\overline{Q}_5$ to be a positive definite matrix.

The conditions (26), (28), (29), (31), and (33) can always be satisfied using a proper selection of $\epsilon$ and $\alpha$. Hence we have demonstrated that $\overline{Q}_1$, $\overline{Q}_2$, $\overline{Q}_3$, $\overline{Q}_4$, $\overline{Q}_5$ are symmetric positive definite matrices, and in consequence $\dot{V}(e_\eta, \nu, \tilde{\eta}, \tilde{\nu})$ in (27) is negative definite out the switching manifold $\tilde{\eta}=0$.

To analyze the asymptotic behavior of the state space solutions at $\tilde{\eta}=0$, consider

$$V(e_\eta, \nu, 0, \tilde{\nu}) = \frac{1}{2}\tilde{\nu}^T P_2\,\tilde{\nu} + \frac{\epsilon}{2}\nu^T M \nu + \frac{\epsilon}{2} e_\eta^T K_p e_\eta + \alpha\tanh(e_\eta)^T K_p J(\eta)\, M\nu,$$

which is a positive definite function because inequality (26). After some calculations and using the Eq. (21), which describes the dynamics of $\tilde{\nu}$ at the switching manifold $\tilde{\eta}=0$, it is possible to show that the time derivative of $V(e_\eta, \nu, 0, \tilde{\nu})$ can be upper bounded as

$$\begin{aligned}\dot{V}(\eta, \nu, 0, \tilde{\nu}) \le & -\lambda_{min}\{Q_a\}\,\|\tilde{\nu}\|^2 \\ & -\epsilon\,\lambda_{min}\{K_v\}\,\|\nu\|^2 + \epsilon\,\overline{k}_1\|\nu\|\,\|\tilde{\nu}\| \\ & +\alpha\lambda_{max}\{K_p\}\lambda_{max}\{K_v\}\,\|\tanh(e_\eta)\|\,\|\nu\| \\ & -\alpha\lambda_{min}\{Q_{e_\eta}\}\,\|\tanh(e_\eta)\|^2 \\ & +\alpha\overline{k}_1\lambda_{max}\{K_p\}\,\|\tanh(e_\eta)\|\,\|\tilde{\nu}\| + \alpha\gamma_1\|\nu\|^2,\end{aligned}$$

where $Q_a = Q_1 + J(\eta)^T K_{\alpha 1} J(\eta)^T$, and $\overline{k}_1$, $Q_{e_\eta}$, and $\gamma_1$ were defined previously. As before, a matrix form can be constructed for the time derivative of $V(\eta, \nu, 0, \tilde{\nu})$; *i.e.*,

$$\begin{aligned}\dot{V}(\eta, \nu, 0, \tilde{\nu}) \le & -\begin{bmatrix}\|\tilde{\nu}\| \\ \|\nu\|\end{bmatrix}^T \overline{Q}_6 \begin{bmatrix}\|\tilde{\nu}\| \\ \|\nu\|\end{bmatrix} \\ & -\begin{bmatrix}\|\tilde{\nu}\| \\ \|\tanh(e_\eta)\|\end{bmatrix}^T \overline{Q}_7 \begin{bmatrix}\|\tilde{\nu}\| \\ \|\tanh(e_\eta)\|\end{bmatrix} \\ & -\begin{bmatrix}\|\nu\| \\ \|\tanh(e_\eta)\|\end{bmatrix}^T \overline{Q}_8 \begin{bmatrix}\|\nu\| \\ \|\tanh(e_\eta)\|\end{bmatrix},\end{aligned}$$

where

$$\overline{Q}_6 = \frac{1}{2}\begin{bmatrix}\lambda_{min}\{Q_a\} & -\epsilon\,\overline{k}_1 \\ -\epsilon\,\overline{k}_1 & \epsilon\,\lambda_{min}\{K_v\} - \alpha\gamma_1\end{bmatrix},$$

$$\overline{Q}_7 = \frac{1}{2}\begin{bmatrix}\epsilon\,\lambda_{min}\{K_v\} - \alpha\gamma_1 & -\alpha\lambda_{max}\{K_p\}\lambda_{max}\{K_v\} \\ -\alpha\lambda_{max}\{K_p\}\lambda_{max}\{K_v\} & \alpha\lambda_{min}\{Q_{e_\eta}\}\end{bmatrix},$$

$$\overline{Q}_8 = \frac{1}{2}\begin{bmatrix}\lambda_{min}\{Q_a\} & -\alpha\,\overline{k}_1\lambda_{max}\{K_p\} \\ -\alpha\,\overline{k}_1\lambda_{max}\{K_p\} & \alpha\,\lambda_{min}\{Q_{e_\eta}\}\end{bmatrix}.$$

Matrix $\overline{Q}_6$ is positive definite if inequality (28),

$$\alpha < \frac{\epsilon\,[\lambda_{min}\{Q_a\}\lambda_{min}\{K_v\} - 2\epsilon\,\overline{k}_1]}{4\gamma_1\lambda_{min}\{Q_a\}},$$

and

$$\epsilon < \frac{\lambda_{min}\{Q_a\}\{K_v\}}{2\overline{k}_1^2},$$

are satisfied. On the other hand, matrix $\overline{Q}_7$ is positive definite by virtue of inequalities (28) and (33). Finally, the condition for $\overline{Q}_8$ to be positive definite is

$$\alpha < \frac{\lambda_{min}\{Q_a\}\lambda_{min}\{Q_{e_\eta}\}}{\overline{k}_1^2 \lambda_{max}\{K_p\}^2}.$$

Thus, positive constants $\alpha$ and $\in$ can always be found such that $\overline{Q}_6$, $\overline{Q}_7$, and $\overline{Q}_8$ are positive definite matrices. Hence the function $\dot{V}(\eta, \nu, 0, \tilde{\nu})$ is negative definite at the switching manifold $\tilde{\eta} = 0$.

Therefore, since $V(e_\eta, \nu, \tilde{\eta}, \tilde{\nu})$ is positive definite and radially unbounded, and its time derivative $\dot{V}(e_\eta, \nu, \tilde{\eta}, \tilde{\nu})$ is negative definite into and out of the switching manifold $\tilde{\eta} = 0$, we conclude that the state space origin of the overall closed-loop system is globally asymptotically stable.

## V. SIMULATIONS RESULTS

Simulations of the proposed scheme using relay observer have been carried out using the following ship model [4]:

$$M = \begin{bmatrix} 1.1274 & 0 & 0 \\ 0 & 1.8902 & -0.0744 \\ 0 & -0.0744 & 0.1274 \end{bmatrix},$$

$$D = \begin{bmatrix} 0.0358 & 0 & 0 \\ 0 & 0.1183 & -0.0124 \\ 0 & -0.0124 & 0.0308 \end{bmatrix},$$

and $K = 0$. This value of $K$ corresponds to a supply vessel (no mooring forces) [1,4]. The numerical simulations where carried out assuming that the ship positions ($x$, $y$, and $\psi$) are corrupted by Gaussian noise with a standard deviation of 0.5m for $x$ and $y$, and 2.5° for $\psi$.

The gains used in the position controller (11)-(13) were

$$K_p = \begin{bmatrix} 20 & 0 & 0 \\ 0 & 20 & 0 \\ 0 & 0 & 10 \end{bmatrix}, \quad K_v = \begin{bmatrix} 10 & 0 & 0 \\ 0 & 10 & 0 \\ 0 & 0 & 5 \end{bmatrix},$$

and

$$K_\eta = \begin{bmatrix} 5 & 0 & 0 \\ 0 & 5 & 0 \\ 0 & 0 & 5 \end{bmatrix}.$$

These gains were obtained using:

$$P_1 = \begin{bmatrix} 3 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad Q_1 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix},$$

and

$$Q_2 = \begin{bmatrix} 0.01 & 0 & 0 \\ 0 & 0.01 & 0 \\ 0 & 0 & 0.01 \end{bmatrix}.$$

For the proposed observer we set

$$K_{\alpha 1} = \begin{bmatrix} 10 & 0 & 0 \\ 0 & 10 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

The gains $K_1$, $K_2(\eta)$, and $K_{\alpha 2}(\eta)$ were computed employing Eqs. (11), (13), and (16). The initial conditions were

$$\eta_1(0) = x(0) = -10\text{m}, \quad \eta_2(0) = y(0) = -10\text{m},$$

and

$$\eta_3(0) = \psi(0) = 30°.$$

The control was programmed with $\eta_d = 0$. The value of $P_2$ is straightforward calculated from (13).

The response of the controller (22)-(23) using the observer proposed in [4], see equations (9)-(10), is shown in Fig. 3. In the other hand, the performance of controller (22)-(23) using the discontinuous observer (14)-(15) is depicted in Fig. 4. The computer simulations shows that the proposed controller with the relay observer has better response than using the observer with the relays terms removed, because the ship positions arrives faster to the desired set-point. Figure 5 compares the time evolution of the control actions for the controller using each observer. Finally, Fig. 6 depicts the performance considering the cumulative energy defined as

$$E = \int_0^t [\tau_1(\sigma)^2 + \tau_2(\sigma)^2 + \tau_3(\sigma)^2]\, d\sigma.$$

From this figure better performance is concluded for the controller using the proposed relay observer.

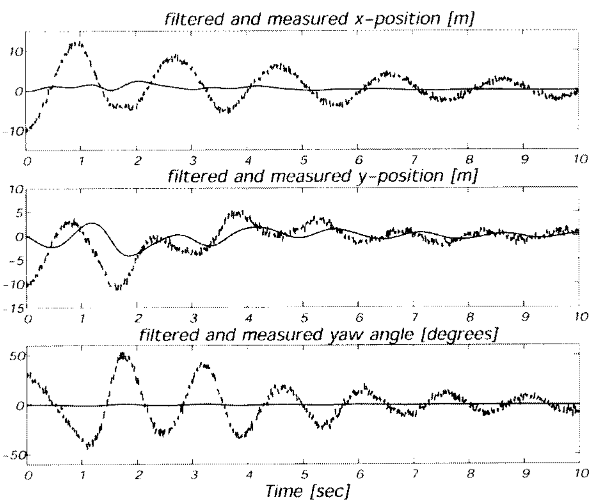


Fig. 3. Simulation results using the controller plus observer with the relays removed: Dotted lines are the noisy position measurements and solid lines are the estimated ship positions.

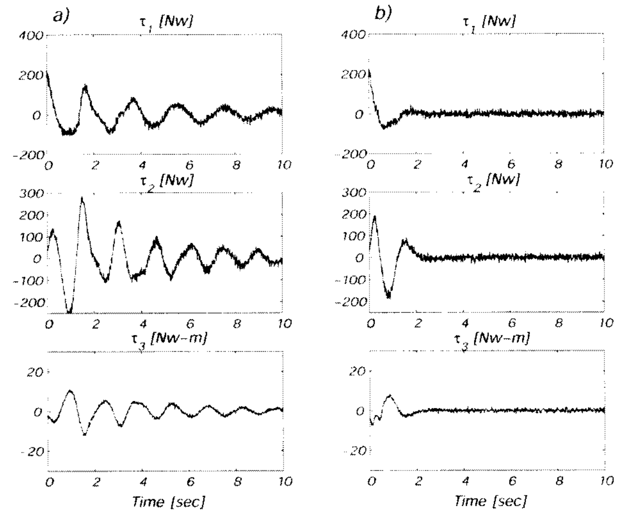


Fig. 5. Performance of the control law: (a) Using the observer with the relay terms removed, (b) using the relay observer.

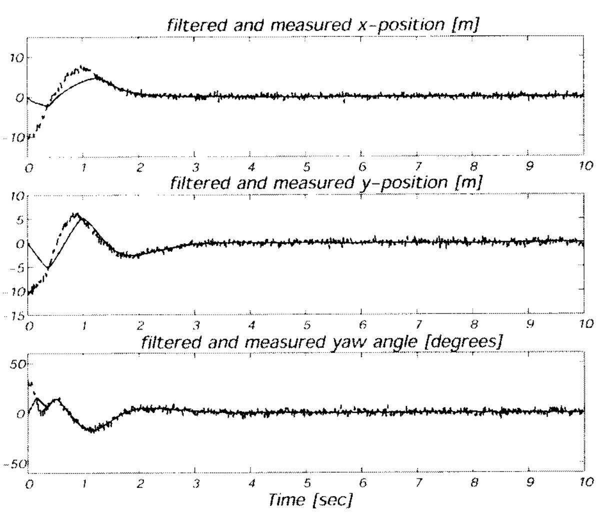


Fig. 4. Simulation results using the controller plus relay observer: Dotted lines are the noisy position measurements and solid lines are the estimated ship positions.

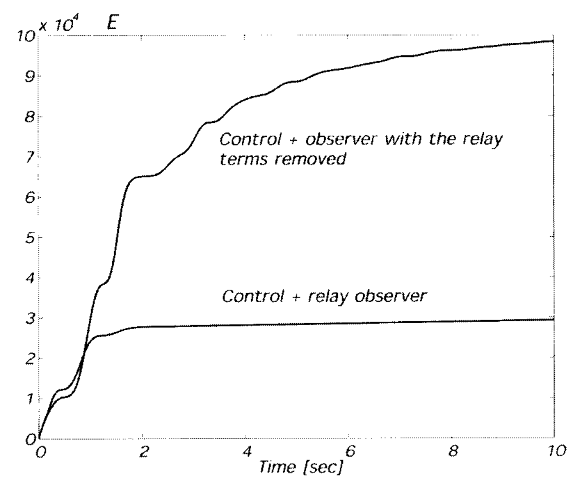


Fig. 6. Plots of $E = \int_0^t [\tau_1(\sigma)^2 + \tau_2(\sigma)^2 + \tau_3(\sigma)^2]\, d\sigma$ for the control law using the observer with relays terms removed and with the relay observer.

## VI. CONCLUDING REMARKS

In this paper we studied the dynamic positioning control of ships using only position measurements. To solve this problem we propose a control law and a relay based observer. In the stability analysis we proposed nonsmooth strict Lyapunov function, concluding asymptotic stability of the closed-loop system equilibrium. By numerical simulations, we compared our results with the proposed in [4], illustrating that the relay-based control system has better performance than the latter.

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