Robust Tracking Control of Servo Systems with Backlash: Nonsmooth $\mathcal{H}_\infty$ Control vs. Linear $\mathcal{H}_\infty$ Control

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Abstract—Backlash and Coulomb friction occurring in systems with mechanical transmissions produce undesired inaccuracies, oscillations, and instability thereby degrading the system performance. In this paper, we made a comparative analysis of a nonsmooth $\mathcal{H}_\infty$ controller [1] vs. a linear $\mathcal{H}_\infty$ controller applied to a two-mass drive system with flexible shaft including backlash and Coulomb friction using measurements of the motor. The numerical results show the effectiveness of the nonsmooth $\mathcal{H}_\infty$ synthesis, whereas the linear $\mathcal{H}_\infty$-controller does not attenuate the undesirable oscillations in the control signals and the plant outputs.

I. INTRODUCTION

Several applications in mechanical systems are affected by different nonlinearities inherent to the systems, such as backlash, dry friction, and elastic bands affecting negatively the system performance, producing oscillations, inaccuracies, and in the worst case, instability of the system. Different techniques are used to reduce the negative effects on the system (see [2], [3]). Some of these techniques include mechanical devices added to the system whereas others use control techniques reducing the undesired effects of the afore-mentioned nonlinearities thus improving the performance of the system.

The control of these mechanisms is more complicated when the full information of the system is not available, specially in the load, which is normally unfeasible. Due to this, it is necessary the use of a filter that would estimate all the states based on the available measurements.

Different works, reported in the literature such as [4], [5], [6], focus on solving the regulation problem. In this paper, the tracking problem of interest is solved using a nonsmooth $\mathcal{H}_\infty$-controller, recently developed in [1], which aims to synthesize a tracking controller, while also attenuating external disturbances. We remark that the nonlinear $\mathcal{H}_\infty$ approach given in [7], [8], and [9] does not admit a straightforward application to nonsmooth systems that is why in our investigation, we follow the nonsmooth $\mathcal{H}_\infty$ synthesis proposed in [10] that allows us to use a nonsmooth representation of the system with backlash.

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Fig. 1. Two-mass system including backlash and Coulomb friction diagram.

The objective of the present work is to extend the nonsmooth $\mathcal{H}_\infty$ synthesis to tracking of servomechanisms with nonsmooth phenomena, involving dry friction and backlash, and with an elastic band, included in the transmission motor-load. The resulting performance using this method is compared with that obtained by using standard linear $\mathcal{H}_\infty$-control. Achieving good performance of the synthesized motion within the present framework is the main contribution of the paper to the existing literature.

The paper is outlined as follows. The dynamic model and the backlash phenomenon of the system are introduced in the next section. In Section III, the nonsmooth output tracking $\mathcal{H}_\infty$-problem is stated, and it is then solved in Section IV using the generic nonsmooth $\mathcal{H}_\infty$ synthesis of [1]. In Section V, performance issues are illustrated in numerical results, comparing the performance of the nonsmooth synthesis with the standard linear $\mathcal{H}_\infty$ controller. Finally, in the last section, some conclusions are presented.

II. DYNAMICAL MODEL

The underlying two-mass servo system with reducer is depicted in Fig. 1 (cf. that of [11], [12]), where the shaft between the two masses is replaced by a flexible shaft, involving backlash, and viscous and Coulomb friction forces, affecting both masses. Thus, the system is exposed to nonsmooth effects that degrade the system performance. The control objective is to make the load to track a desired trajectory provided that only the measurement of motor position is available.

The dynamics of the system are governed by

$$
J_1 \ddot{\theta}_1 + c_1 \dot{\theta}_1 + f_1 \text{sign}(\dot{\theta}_1) + T(\Delta_\theta) = \tau + w_1
$$
$$
J_2 \ddot{\theta}_2 + c_2 \dot{\theta}_2 + f_2 \text{sign}(\dot{\theta}_2) = n_g T(\Delta_\theta) + w_2
$$

where $\theta_1, \theta_2 \in \mathbb{R}$ denote the position of the motor and the position of the load. $J_1, J_2, f_1, f_2, c_1, c_2 \in \mathbb{R}^+$ are the inertia, the dry friction and the viscous friction of the motor and that of the load, respectively. The input torque
\( \tau \) represents the control action, \( T \) is referred to as the transmitted torque from the motor to the load, and \( n_g \in \mathbb{R}^+ \) is the reducer ratio. Finally, \( u_1, w_2 \in L_2 \) are introduced to take into account different perturbations that affect the motor and the load, respectively.

We represent the transmitted torque \( T(\Delta_\theta) \), through a backlash model with an amplitude \( j \in \mathbb{R}^+ \), using the following dead-zone model [13]

\[
T(\Delta_\theta) = \begin{cases} 
k_g^{-2} (\Delta_\theta - j \text{sign}(\Delta_\theta)), & \text{if } |\Delta_\theta| \leq j, \\
0, & \text{otherwise}
\end{cases}
\]

where \( k \) is the stiffness the elastic bound and

\[
\Delta_\theta = \theta_1 - n_g \theta_2.
\]

This model is depicted in Fig. 2 (solid line). The above model can be rewritten as

\[
T(\Delta_\theta) = T_1(\Delta_\theta) + T_2(\Delta_\theta)
\]

in order to separate the linear term \( T_1 \) of the nonlinear term \( T_2 \) where

\[
T_1(\Delta_\theta) = k_n^{-2} \Delta_\theta,
\]

\[
T_2(\Delta_\theta) = \begin{cases} 
-k_n^{-2} \Delta_\theta, & \text{if } |\Delta_\theta| \leq j, \\
-k_n^{-2} j \text{sign}(\Delta_\theta), & \text{otherwise}.
\end{cases}
\]

The above two functions \( T_1 \) and \( T_2 \) are depicted in Fig. 2.

### III. PROBLEM STATEMENT

The problem of interest is to make the load to asymptotically track a sufficiently smooth reference signal \( \theta_2^n(t) \) such that

\[
\dot{\theta}_2^n(t) \neq 0 \quad \text{for almost all } t,
\]

while also attenuating the system disturbances and the measurement error \( w_y \in \mathbb{R} \). The imperfect motor position measurement

\[
y = \theta_1 + w_y
\]

is assumed to be the only available information on the state of the system.

To formally state the problem, let us introduce the state deviation vector \( x \in \mathbb{R}^4 \) with components

\[
x_1 = \theta_1 - \theta_2^n(t),
\]

\[
x_2 = \dot{\theta}_1 - \dot{\theta}_2^n(t),
\]

\[
x_3 = \theta_2 - \theta_2^n,
\]

\[
x_4 = \dot{\theta}_2 - \dot{\theta}_2^n
\]

where the reference motor signal \( \theta_2^n(t) \) is deliberately specified in the form

\[
\theta_2^n(t) = n_g \theta_2^n + \text{sign}(\kappa)(j + k^{-1} n_g |\kappa|),
\]

with

\[
\kappa = c_2 \dot{\theta}_2^n + f_2 \text{sign}(\dot{\theta}_2^n) + J_2 \dot{\theta}_2^n
\]

to match the reference load signal \( \theta_2^n(t) \) (see our further comment on the verification of Assumption A2).

Then in terms of the state deviation \( x \in \mathbb{R}^4 \), the plant dynamics (1) are governed by the time-varying system

\[
\dot{x} = \begin{bmatrix}
x_2 \\
x_4
\end{bmatrix}
-
\begin{bmatrix}
0 \\
0
\end{bmatrix}
+
\begin{bmatrix}
J_1^{-1}(-c_1(x_2 + \dot{\theta}_1^n) - f_1 \text{sign}(x_2 + \dot{\theta}_1^n)) \\
J_2^{-1}(-c_2(x_4 + \dot{\theta}_2^n) - f_2 \text{sign}(x_4 + \dot{\theta}_2^n))
\end{bmatrix}
\]

where \( T(\Delta_x) \) is given by (2) with

\[
\Delta_x = x_1 + \theta_1^n - n_g (x_3 + \theta_2^n).
\]

Let the output to be controlled be given

\[
z = \begin{bmatrix} u \\ \rho x \end{bmatrix}
\]

with \( \rho = \text{diag}\{\rho_i\}, \rho_i \geq 0, i = 1, 2, 3, 4 \) and \( \|\rho\| > 0 \) whereas the available measurement be the motor position, corrupted by \( w_y \in \mathbb{R} \), this is

\[
y = x_1 + \theta_1^n + w_y.
\]

The \( H_\infty \)-control problem in question is stated as follows. Given the error system (12)–(15) and real number \( \gamma > 0 \), it is required to find (if any) a causal dynamic output feedback compensator

\[
u = K(\xi, t),
\]

\[\dot{\xi} = F(\xi, y, t),\]

with internal state \( \xi \in \mathbb{R}^4 \) such that the closed-loop system is internally uniformly asymptotically stable around the origin whereas its \( L_2 \)-gain is locally less than \( \gamma \). For convenience of the reader recall that the underlying system locally possesses \( L_2 \)-gain less than \( \gamma \) iff the following inequality holds

\[
\int_0^T \|z(t)\|^2 dt < \gamma^2 \int_0^T \|w(t)\|^2 dt.
\]
for all $T > 0$, for all the system trajectories initialized in the origin, and for all piecewise continuous function $w(t) \in L_2$ such that the state trajectories remain in a vicinity of the origin.

IV. GENERIC $H_\infty$ SYNTHESIS

The above nonsmooth $H_\infty$ problem is subsequently solved within the framework proposed in [10].

Let us consider a nonautonomous nonlinear system of the form

$$\dot{x}(t) = f(x(t), t) + g_1(x(t), t)w(t) + g_2(x(t), t)u(t),$$
$$z(t) = h_1(x(t), t) + k_{12}(x(t), t),$$
$$y(t) = h_2(x(t), t) + k_{21}(x(t), t)u(t),$$

(18)

where $x(t) \in \mathbb{R}^n$ is the state vector, $u(t) \in \mathbb{R}^m$ is the control input, $w(t) \in \mathbb{R}^r$ is the unknown disturbance, $z(t) \in \mathbb{R}^l$ is the output to be controlled, $y(t) \in \mathbb{R}^p$ is the available measurement of the system. In the present work, $x$ represents the solution of system (12), $w = [w_1 \ w_m \ w_z] \mathbb{R}^5$, $z \in \mathbb{R}^5$ is given by (14), and the control input

$$\tau = u + u_c,$$

(19)

is composed of the reference compensator

$$u_c(x) = J_1\theta_1^{1} + c_1\theta_1^{1} + f_1\text{sign}(\theta_1^{1}) + k_n^{-1}2\theta_1^{1} - k_n^{-1}\theta_2^{1} + T_g(\Delta_y),$$

(20)

and the stabilizing control component $u$, attenuating external disturbances. Hence, system (18) is specified with

$$f(x, t) = \begin{bmatrix} J_1^{-1}(-k_n^{-2}x_1 - c_1x_2 + k_n^{-1}x_3 + \psi_1) \\ J_2^{-1}(k_n^{-1}x_1 - k_nx_3 - c_2x_4 + \psi_2) \end{bmatrix},$$
$$g_1(x) = \begin{bmatrix} J_1^{-1} \\ 0 \end{bmatrix}, \quad g_2(x) = \begin{bmatrix} 0 \\ J_2^{-1} \end{bmatrix},$$
$$h_1(x, t) = \begin{bmatrix} \rho_1x_1 \\ \rho_2x_2 \\ \rho_3x_3 \\ \rho_4x_4 \\ \rho_5x_5 \end{bmatrix}, \quad k_{12}(x) = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix},$$
$$h_2(x, t) = x_1 + \theta_1^{1}, \quad k_{21}(x) = \begin{bmatrix} 0 \ 0 \ 0 \end{bmatrix},$$

(22)

where

$$\psi_1 = -f_1\text{sign}(x_2) + f_1\text{sign}(\theta_1^{1}) - T_2(\Delta_x) + T_g(\Delta_y),$$
$$\psi_2 = -J_2\theta_2^{1} - c_2\theta_2^{1} - k\theta_2^{1} + k_n^{-1}\theta_1^{1} - f_2\text{sign}(x_4) + \theta_2^{1} + n_2T_2(\Delta_x).$$

(23)

The following assumptions are imposed on the generic system (18):

A1. The functions $f(x, t)$, $g_1(x, t)$, $g_2(x, t)$, $h_1(x, t)$, $h_2(x, t)$, $k_{12}(x, t)$, $k_{21}(x, t)$ are continuous and locally Lipschitz continuous in $x$ for all $t$;

A2. $f(0, t) = 0, h_1(0, t) = 0, h_2(0, t) = 0$ for all $t$;

A3. $h_2^T(x, t)k_{12}(x, t) = 0, k_{12}^T(x, t)k_{12}(x, t) = I,$

$k_{21}(x, t)g_2^T(x, t) = 0, k_{21}(x, t)k_{21}(x, t) = I$ for all $t$.

A4. For almost all $t \in \mathbb{R}$, there exists a neighborhood $U_t(0)$ of the origin $x = 0$, possibly dependent on $t$, such that the functions listed in Assumption A1, are uniformly bounded in $t$, twice continuously differentiable in $x$, and their first and second order derivatives are piecewise continuous and uniformly bounded in $t \in \mathbb{R}$ for all $x \in U_t(0)$.

Assumption A1 admits nonsmooth nonlinearities guaranteeing the well-posedness of the above dynamical system under integrable exogenous inputs.

Assumption A2 ensures that the origin is an equilibrium point of the non-driven ($u = 0$) disturbance-free ($w = 0$) dynamical system (18). To fulfill assumption A2 the reference motor signal $\theta_1^{1}$ has been pre-specified in the form (10), (11) that makes $f(0, t) = 0$ for almost all $t$, holding the condition $\theta_1^{1} - n_2\theta_2^{1} > j$, i.e., the system intends to keep the involved mechanisms always in contact under the backlash effect.

Assumption A3 is a simplifying assumption inherited from the standard $H_\infty$-control problem.

Assumption A4, made for a technical reason, allows one to locally linearize the closed-loop system, driven by a smooth feedback controller.

The nonsmooth $H_\infty$-control problem for a generic system (18) is to find a locally stabilizing output feedback controller (16), under the available measurement $y$, such that the $L_2$-gain of the closed-loop system (18), driven by (16), is locally less than $\gamma$. Solving the above problem under $\gamma$ approaching the infimal achievable level $\gamma^*$ in (17) yields a (sub)optimal $H_\infty$-controller with the (sub)optimal disturbance attenuation level $\gamma^*$ ($\gamma > \gamma^*$).

The local synthesis involves the standard linear $H_\infty$-control problem for the linearized system

$$\dot{x}(t) = A(t)x(t) + B_1(t)w(t) + B_2(t)u(t),$$
$$z(t) = C_1(t)x(t) + D_{12}(t)u(t),$$
$$y(t) = C_2(t)x(t) + D_{21}(t)w(t),$$

(25)

where

$$A(t) = \frac{\partial f}{\partial x}(0, t), \quad B_1(t) = g_1(0, t), \quad B_2(t) = g_2(0, t),$$
$$C_1(t) = \frac{\partial h_1}{\partial x}(0, t), \quad C_2(t) = \frac{\partial h_2}{\partial x}(0, t),$$
$$D_{12}(t) = k_{12}(0, t), \quad D_{21}(t) = k_{21}(0, t).$$

(26)

Using the time-varying version [10] of the real bounded lemma [14], one can note that the following conditions are necessary and sufficient for a solution of the linear $H_\infty$-control problem to exist:
there exists a positive constant \( \varepsilon_0 \) such that the perturbed Riccati equation

\[
-\dot{P}_\varepsilon = P_\varepsilon(t)A(t) + A^T(t)P_\varepsilon(t) + C_1^T(t)C_1(t) + \frac{1}{\gamma^2} B_1 B_1^T \varepsilon - B_2 B_2^T (t) P_\varepsilon(t) + \varepsilon I,
\]

possesses a positive definite symmetric solution \( P_\varepsilon(t) \) for each \( \varepsilon \in (0, \varepsilon_0) \);

C2. while being coupled to (27), the perturbed Riccati equation

\[
\dot{Z}_\varepsilon = A_2(t) Z_\varepsilon(t) + Z_\varepsilon(t) A_2^T(t) + B_1(t) B_1^T(t) + \frac{1}{\gamma^2} P_\varepsilon(t) B_2 B_2^T P_\varepsilon(t) C_2^T \varepsilon C_2(t) Z_\varepsilon(t) + \varepsilon I
\]

possesses a positive definite symmetric solution \( Z_\varepsilon(t) \) with \( A_2(t) = A(t) + \frac{1}{\gamma^2} B_1 B_1^T(t) P_\varepsilon(t) \).

Equations (27) and (28) are now utilized to derive a local solution of the nonsmooth \( \mathcal{H}_\infty \)-control problem for system (18). Under partial state measurements, the local \( \mathcal{H}_\infty \) synthesis is augmented with a dynamic compensator running in parallel. The compensator is derived by means of the perturbed Riccati equations (27) and (28), that appear in solving the \( \mathcal{H}_\infty \)-control problem for the linear system (25) when these equations have uniformly bounded positive definite solutions. The following result is inherited from [1].

**Theorem 1:** Consider system (18) with assumptions A1-A4. Let conditions C1 and C2 be satisfied with a certain \( \gamma > 0 \) and let \( (P_\varepsilon(t), Z_\varepsilon(t)) \) be a uniformly bounded positive definite symmetric solution of (27) and (28) under some \( \varepsilon > 0 \). Then, the causal dynamic output feedback compensator

\[
\dot{\xi} = f(\xi, t) + \left[ \frac{1}{\gamma^2} g_1(\xi, t) g_1^T(\xi, t) - g_2(\xi, t) g_2^T(\xi, t) \right] P_\varepsilon(t) \xi + Z_\varepsilon(t) C_2^T(t) [y(t) - h_2(\xi, t)]
\]

\[
u = -g_2^T(\xi, t) P_\varepsilon(t) \xi
\]

is a local solution of the \( \mathcal{H}_\infty \)-control problem with the disturbance attenuation level \( \gamma \).

**Proof:** Theorem 1 addresses nonsmooth systems which are smooth in the origin as opposed to [10] (Theorem 24) where the underlying nonsmooth system is not necessarily smooth in the origin. Thus, the validation of Theorem 1 is confined to the specification of [10, Thm. 24] to the present case. The detailed proof appears elsewhere [1].

### A. Linear \( \mathcal{H}_\infty \) Control

The next aim is to compare the performance of the controller described above with the standard linear \( \mathcal{H}_\infty \)-control, which has been widely discussed in the literature. To address the standard linear \( \mathcal{H}_\infty \)-control problem, the simplifying assumptions are additionally imposed on the linearized system (25):

A5. \( (A, B_1) \) is stabilizable and \( (C_1, A) \) is detectable.

A6. \( (A, B_2) \) is stabilizable and \( (C_2, A) \) is detectable.

### V. NUMERICAL RESULTS

To illustrate the theory the nonsmooth \( \mathcal{H}_\infty \) approach, developed to the underlying system depicted in Fig. 1, is applied to Model 220 Industrial Plant Emulator from Educational Control Products (ECP). Fig. 3 shows the Industrial Plant Emulator whose diagram is equivalent to that shown in Fig. 1. The system parameter values are specified in Tab. I. These values were obtained from [1] and using the manufacturer information [15].

Fig. 4 shows the effects of the backlash, friction and flexibility in the transmission bands that occur in the system.
enforced by a control input of sine type. These nonlinearities are a real challenge in performing an efficient control.

The efficacy of the proposed solution to the tracking problem is illustrated with numerical results. The controller parameters used in the numerical simulations were $\gamma = 80$, $\rho = \text{diag}(1, 0, 20, 0)$, $\varepsilon = 0.001$, and we applied the disturbance $w = [0.15 \cos(0.4\pi t) \quad 0.1 \cos(0.4\pi t)]$. The reference load signal is specified by $\theta_1^r = 5\sin(0.2\pi t)$. Additionally, we consider parametrical variations in the controller design, taking into account the following parameter values: $J_1 = 0.003$, $J_2 = 0.0351$, $c_1 = 0.00028$, $c_2 = 0.0048$, $f_1 = 0.0252$, $f_2 = 0.338$, and $k = 32.96$. The resulting trajectories and applied torque when the nonsmooth controller is applied are depicted in Figs. 5 and 6. These figures demonstrate that the nonsmooth controller achieve that the load position $\theta_2$ tracks the reference load signal $\theta_1^r$ and attenuates external disturbances including parametrical variations.

Using same parameters as for the nonsmooth controller, the resulting trajectories and applied torque for the linear controller are depicted in Figs. 7 and 8. These figures show that the linear controller is unable to achieve that the load position $\theta_2$ tracks efficiently the desirable trajectory $\theta_1^r$, producing undesired oscillations in the control signal and the plant outputs, leading the system to an unacceptable behavior.

VI. CONCLUSIONS

In this paper, the nonsmooth $H_\infty$ tracking is developed to a servomechanism with backlash, dry friction and elastic bands in the presence of external disturbances and parametrical variations. In spite of the presence of nonsmooth phenomena, the proposed nonsmooth $H_\infty$ controller asymptotically tracks a reference load signal using motor position measurements only while also attenuating external disturbances. Robustness features of the proposed nonsmooth synthesis are additionally supported in the numerical study,
Fig. 9. Performance comparison between the nonsmooth $H_\infty$ control and standard linear $H_\infty$ control.

made for an industrial emulator. The numerical results show the superiority of the nonsmooth $H_\infty$ controller over the standard linear $H_\infty$ controller which unavoidably produces undesired oscillations in the control signal and the plant outputs.

REFERENCES


